

Dimers and analytic torsion I

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Abstract

In the dimer model, a configuration consists of a perfect matching of a fixed graph. If the underlying graph is planar and bipartite, such a configuration is associated to a height function. For appropriate “critical” (weighted) graphs, this height function is known to converge in the fine mesh limit to a Gaussian free field, following in particular Kenyon’s work.

In the present article, we study the asymptotics of smoothed and local field observables from the point of view of families of Cauchy-Riemann operators and their determinants. This allows in particular to obtain a functional invariance principle for the field; characterise completely the limiting field on toroidal graphs as a compactified free field; analyse electric correlators; and settle the Fisher-Stephenson conjecture on monomer correlators.

The analysis is based on comparing the variation of determinants of families of (continuous) CR operators with that of their discrete (finite dimensional) approximants. This relies in turn on estimating precisely inverting kernels, in particular near singularities. In order to treat correlators of “singular” local operators, elements of (multiplicatively) multi-valued discrete holomorphic functions are discussed.

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1 Introduction

The dimer model is a classical model of statistical mechanics; it consists in sampling uniformly (or with weights) perfect matchings of a bipartite graph. For planar graphs, Kasteleyn showed that the partition function can be expressed as the Pfaffian of a properly signed (weighted) adjacency matrix for the graph. In the small mesh limit, this operator can be interpreted as a finite difference version of a $\bar{\partial}$ -operator. In the “Temperleyan” case we will be considering, dimer configurations are in measure-preserving bijection with spanning trees on a related graph.

Following Thurston, one may associate a height function to a dimer configuration on the square lattice (or another bipartite graph). As suggested by Benjamini, this height function can be understood in terms of windings of the associated spanning tree.

In [26], Kenyon considers the scaling limit of the height function of dimers on the square lattice for planar domains with appropriate boundary conditions, and proves that in the scaling limit, it converges in distribution to a conformally invariant object for n -connected domains. This is based on precise asymptotics of the inverse Kasteleyn matrix. In [27], he identifies this limiting distribution as the classical massless Gaussian free field, in the simply connected case.

In the present article, we are interested in several extensions of these results. The main point of view is that the invariance principles under consideration can be related in a natural way to Quillen’s theory of families of Cauchy-Riemann operators ([38]).

We outline the general strategy, first in the case of dimer tilings of the plane. Let h be the (discrete, random) height field and ϕ a real compactly supported smooth test function. Then we shall see that

$$\mathbb{E}(\exp(-i \int h \Delta \phi)) = \frac{\det(K_{\bar{\partial}\phi})}{\det(K)}(\dots)$$

where K is a finite difference analogue of $\bar{\partial}$, while $K_{\bar{\partial}\phi}$ is an analogue of the Cauchy-Riemann operator $\bar{\partial} + (\bar{\partial}\phi)$, and (\dots) is a multiplicative correction which depends on ϕ . Here, h is piecewise constant on faces of the graph carrying the perfect matching.

In order to take limits, it is convenient to consider Laplacian-type operators:

$$\mathbb{E}(\exp(-i \int h \Delta \phi))^2 = \frac{\det((K_{\bar{\partial}\phi})^* K_{\bar{\partial}\phi})}{\det(K^* K)}(\dots)$$

In the continuum, there is a standard way to associate a “determinant” to such operators, namely ζ -regularisation, introduced by Ray and Singer ([39]). Let us be optimistic and assume for now that in the scaling limit, we can relate the above quantity to

$$\frac{\det_{\zeta}(\Delta_{\bar{\partial}\phi})}{\det_{\zeta}(\Delta)}$$

where $\Delta = \bar{\partial}^* \bar{\partial}$, $\Delta_{\alpha} = (\bar{\partial} + \alpha)^* (\bar{\partial} + \alpha)$. Then (a version of) the Quillen curvature formula ([38]) yields:

$$\partial_{tt} \log(\det_{\zeta} \Delta_{\bar{\partial}\phi}) = -\frac{i}{2\pi} \int (\partial\phi) \wedge (\bar{\partial}\phi) = -\frac{1}{4\pi} \int \|\nabla\phi\|^2$$

(since ϕ is real). This is consistent with the defining property of the (normalised) massless free field:

$$\mathbb{E}_{FF}(\exp(it \int h \Delta\phi)) = \exp(-\frac{\sigma^2 t^2}{2} \int \|\nabla\phi\|^2)$$

where σ is a scale constant.

Now let us briefly discuss the situation for more complex topologies, in particular on a torus Σ . As shown by Kasteleyn, enumeration of dimer configurations by determinants is more complicated, involving spinor bundles. Also, the height function is only well-defined as a closed 1-form dh (alternatively, as an additively multiply-valued function h). The Hodge decomposition reads:

$$dh = \omega_h + dh_0$$

where ω_h is a closed harmonic form (hence lying in a finite dimensional vector space $\simeq H^1(\Sigma)$; actually on a lattice Λ of this space due to model constraints) and h_0 is a single-valued function (defined up to an additive constant). We are interested in the limiting joint distribution of these objects. It is specified by the characteristic functional $\alpha \mapsto \mathbb{E}(i\Re \int dh \wedge \alpha)$, where α ranges over $(0,1)$ -forms (with associated Cauchy-Riemann operator $\bar{\partial} + \alpha$). The $\bar{\partial}$ -Hodge decomposition gives $\alpha = \bar{\omega} + \bar{\partial}\phi$ where ω is an abelian form (a holomorphic 1-form). Now:

$$\int dh \wedge \alpha = \int \omega_h \wedge \bar{\omega} + \int dh_0 \wedge \bar{\partial}\phi$$

Indeed, $\int \bar{\omega} \wedge \partial h_0 = -\int (\partial \bar{\omega}) h_0 = 0$ and $\int (\bar{\partial}\phi) \wedge \omega_h = \frac{1}{2} \int (\bar{\partial}\phi) \wedge (\omega_h + i * \omega_h) = \frac{1}{2} \int (d\phi) \wedge (\omega_h + i * \omega_h) = 0$ since $d\omega_h = d * \omega_h = 0$. The analysis will again be based on relating the family of CR operators $\alpha \mapsto \bar{\partial} + \alpha$ to its discrete counterpart, up to the level of (ζ) -determinants.

Beyond these “smooth” observables, for which convergence should be quite robust, stricter conditions on the underlying graph (periodicity or isoradiality) allow to control “vertex” observables, such as electric correlators of type $\mathbb{E}(\exp(i \sum_j \alpha_j h(z_j)))$ and monomer correlations (as in the Fisher-Stephenson conjecture). This will be addressed by identifying suitable families of CR operators and controlling the convergence of their finite difference approximations.

New results obtained in this article include: functional convergence to the compactified free field for dimers on toroidal graphs; asymptotics of electric vertex correlators; and the Fisher-Stephenson conjecture for monomer correlations. In a follow-up article, extensions of these results, in particular to domains with boundary (bounded simply and multiply connected domains), will be considered.

The article is organised as follows. Section 2 provides brief background on the continuous free field, the discrete dimer height field, and families of CR operators. In Section 3, we review the necessary results on isoradial dimers. Invariance principles in the plane are discussed in Section 4. The case of toroidal graphs is treated in Section 5. Section 6 describes a general surgery principle for (converging sequences) of discrete CR operators. Discrete multi-valued holomorphic functions and electric correlators are described in Section 7. Monomers correlators are studied in Section 8.

2 Background and overview

The main goal of this article is to express asymptotics of natural dimer observables in terms of a (compactified) free field; a key tool will be the study of families of (discrete and continuous) Cauchy-Riemann operators and their variational properties. In this section we provide a brief introduction to these objects.

2.1 Free field

The Gaussian (or massless) free field is a Gaussian field with covariance kernel given by the Laplacian Green's function (or a multiple thereof). For a general introduction to the free field, see eg [40, 19, 14]. Here we will consider the free field on Σ , where Σ is the plane or a torus $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. In both cases, the Laplacian has a non-trivial kernel consisting of constant functions. Consequently, the free field is only defined there up to an additive constant.

Following Gross' abstract Wiener space approach, one may regard the free field as a random element ϕ of a Banach space, usually a Sobolev space modulo additive constants: $H^{-\varepsilon}(\Sigma)/\mathbb{R}$ (for some positive ε). Note that elements of this space, and thus realisations of the free field, are distributions. In order to avoid quotienting by constant functions, one may consider its distributional total derivative, the *current* $J = d\phi$.

The (centered) free field is characterised by the fact that for any zero-mean test function $\psi \in C_c^\infty(\Sigma)$, $\int_\Sigma (\psi\phi)dA$ is a centered Gaussian variable with variance $g \int_\Sigma \psi \Delta^{-1} \psi dA$, where $g > 0$ is a scaling constant (here Δ is the positive Laplacian). In other words,

$$\langle \exp(i \int_\Sigma (\psi\phi)dA) \rangle = \exp(-\frac{g}{2} \int_\Sigma \psi \Delta^{-1} \psi dA)$$

where $\langle \rangle$ denotes the expectation with respect to the free field. Instead of integrating against a smooth test function, it is often useful and natural to consider local “operators”, involving the local behaviour of the field at a certain number of selected points. For instance, one may consider current correlations, given by the Wick formula:

$$\langle J(z_1) \dots J(z_{2n}) \rangle = g^n \sum_{\{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}} \prod_{i=1}^n d_{z_{\alpha_i}} d_{z_{\beta_i}} G_\Sigma(z_{\alpha_i}, z_{\beta_i})$$

where the sum bears on the pairings $\{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$ of indices in $\{1, \dots, 2n\}$ (for an odd number of currents, the correlator vanishes due to the symmetry $\phi \leftrightarrow -\phi$). This expression makes sense when integrated by regular enough test functions $\psi_1(z_1), \dots, \psi_{2n}(z_{2n})$.

After distributional derivatives of the field, the next local operators to consider are *electric* vertex operators ([18, 13]), which we write formally as $\exp(i\alpha\phi(z))$ where the constant α is the *charge*. The difficulty is that, as ϕ is not defined pointwise but as a distribution, it may not *a priori* be composed with the nonlinear function \exp . A standard normalisation scheme for the correlator “ $\langle \exp(\sum_j \alpha_j \phi(z_j)) \rangle$ ” consists in replacing $\exp(i\alpha\phi(z))$ with the (well-defined) $\exp(i\alpha(\pi\delta^2)^{-1} \int_{D(z, \delta)} \phi(z) dA(z))$, letting δ go to zero, and discarding the leading multiplicative term $\exp(-\frac{g}{2} \sum_j \alpha_j^2 (\pi\delta^2)^{-2} \int_{D(0, \delta)^2} G_\mathbb{C}(x, y) dA(x) dA(y))$, which reflects the local blow-up of variance and not the geometry of the domain or the long range correlations. Alternatively, if $\tilde{\phi}$ is a field with continuous realisations and covariance kernel C , we have:

$$\langle \exp(i \sum_j \alpha_j \tilde{\phi}(z_j)) \rangle = \exp(-\frac{1}{2} \sum_{j, k} \alpha_j \alpha_k C(z_j, z_k))$$

Here, the covariance kernel gG_Σ blows up on the diagonal; to obtain a finite regularised correlator, we replace the coincident Green kernel “ $G(x, x)$ ” with the Robin kernel: $\tilde{G}_\Sigma(x, x) = r_\Sigma(x) = \lim_{y \rightarrow x} G_\Sigma(x, y) -$

$\frac{1}{2\pi} \log |x - y|$ (otherwise $\tilde{G}_\Sigma(x, y) = G_\Sigma(x, y)$) to obtain the expression:

$$\langle : \exp(i \sum_j \alpha_j \phi(z_j)) : \rangle = \exp(-\frac{g}{2} \sum_{j,k} \tilde{G}_\Sigma(z_j, z_k))$$

where the colons recall the regularisation scheme, and we need $\sum_j \alpha_j = 0$. In the plane, this evaluates to the elementary expression:

$$\langle : \exp(i \sum_j \alpha_j \phi(z_j)) : \rangle_{\mathbb{C}} = \prod_{j < k} |z_j - z_k|^{\frac{g}{2\pi} \alpha_j \alpha_k}$$

Another set of “local operators” is given by *magnetic* correlators. These are disorder variables, ie they represent a modification of the state space rather than a modification of the state weights. Specifically, given marked points z_1, \dots, z_n and weights $\alpha_1, \dots, \alpha_n$, one considers additively multivalued functions which increase by α_i when cycling clockwise around z_i ; one requires $\sum_j \alpha_j = 0$ so that the function is single-valued near infinity. This defines an affine state space with a single element of minimal (at least after normalisation) Dirichlet energy:

$$\phi_0 = \frac{1}{2\pi} \sum_j \alpha_j \Im \log(z - z_j)$$

In the Gaussian formalism, offsetting fields by a harmonic function ϕ_0 results in multiplying the partition function by $\exp(-\frac{g^{-1}}{2} \int_\Sigma |\nabla \phi_0|^2 dA)$ (see eg [14] for a discussion of free field partition functions). Here the Dirichlet energy is infinite, due to logarithmic singularities at the z_j 's. A regularised Dirichlet energy is obtained by discarding the leading part of $\int_{\Sigma \setminus (\cup_j D(z_j, \delta))} |\nabla \phi_0|^2 dA$. Representing by $\mathcal{O}_\alpha(z)$ a magnetic charge α at z , we will consider the following regularised correlator:

$$\langle : \prod_j \mathcal{O}_{\alpha_j}(z_j) : \rangle = \exp\left(-\frac{g^{-1}}{2} \int_\Sigma^{reg} |\nabla \phi_0|^2 dA\right)$$

where

$$\int_\Sigma^{reg} |\nabla \phi_0|^2 dA = \lim_{\delta \searrow 0} \left(\int_{\Sigma \setminus (\cup_j D(z_j, \delta))} |\nabla \phi_0|^2 dA + \frac{1}{2\pi} \sum_j \alpha_j^2 \log |\delta| \right)$$

Again when $\Sigma = \mathbb{C}$ this correlator has a simple evaluation. Indeed, note that ϕ_0 has the same Dirichlet energy as its (single valued) harmonic conjugate $\tilde{\phi}_0 = \sum_j \alpha_j \Re \log(z - z_j)$. By Green's formula,

$$\begin{aligned} \int_{\Sigma \setminus (\cup_j D(z_j, \delta))} |\nabla \tilde{\phi}_0|^2 dA &= \sum_j \int_{C(z_j, \delta)} \tilde{\phi}_0 \partial_n \tilde{\phi}_0 d\ell = -\frac{1}{2\pi} \sum_j (\alpha_j + O(\delta)) (\alpha_j \log |\delta| + \sum_{k \neq j} \alpha_k \log |z_k - z_j| + O(\delta)) \\ &= -\frac{1}{2\pi} \left(\sum_j \alpha_j^2 \log |\delta| - \frac{1}{2\pi} \sum_{j \neq k} \alpha_j \alpha_k \log |z_j - z_k| \right) \end{aligned}$$

so that

$$\langle : \prod_j \mathcal{O}_{\alpha_j}(z_j) : \rangle_{\mathbb{C}} = \prod_{j < k} |z_j - z_k|^{\frac{g^{-1}}{2\pi} \alpha_j \alpha_k}$$

Up to now, we have discussed the *scalar* free field, taking real values. An important ([18, 13]) variant is the *compactified* free field, taking values in the circle $\mathbb{R}/r\mathbb{Z}$; let us begin with an informal discussion. Given two Riemannian manifolds S and T , the classical harmonic mapping problem consists in finding a mapping $\phi : S \rightarrow T$ which minimises Dirichlet energy, for instance within a homotopy class. For example, the harmonic mappings $\mathbb{C}/L \rightarrow \mathbb{R}/r\mathbb{Z}$ are written $z \mapsto \Re(z\bar{w})$, where $\Re(\ell\bar{w}) \in r\mathbb{Z}$ for all $\ell \in L$ (L a lattice). In the quantised version of the problem, one consider mappings $\phi : \Sigma \rightarrow \mathbb{R}/r\mathbb{Z}$ (Σ a surface) with action functional given by the Dirichlet energy

$$S(\phi) = \frac{g^{-1}}{2} \int_\Sigma |\nabla \phi|^2 dVol$$

and distribution formally given by $e^{-S(\phi)}\mathcal{D}\phi$. Observe that ϕ can be decomposed as the sum of a harmonic mapping ϕ_h and a scalar part $\phi_s : \Sigma \rightarrow \mathbb{R}$, projected on $\mathbb{R}/r\mathbb{Z}$; moreover the decomposition is unique up to additive constants. In terms of current, it amounts to considering the Hodge decomposition of $J = d\phi$:

$$J = d\phi = \omega_h + dh_0$$

where ω_h is a closed harmonic form (the *instanton* component) and h_0 is a scalar field, well defined up to additive constant. Moreover, the action functional splits:

$$S(\phi) = \frac{g^{-1}}{2} \left(\int_{\Sigma} \omega_h \wedge *\omega_h + \int_{\Sigma} |\nabla h_0|^2 dVol \right)$$

It is thus natural to introduce a free field on Σ with compactification radius r as the data (ω_h, h_0) where the two components are independent. The 1-form ω_h has unnormalised distribution

$$\exp\left(-\frac{g^{-1}}{2} \int_{\Sigma} \omega_h \wedge *\omega_h\right) d\mu(\omega_h)$$

where μ is the counting measure on the lattice of harmonic 1-forms with periods integer multiple of r : $\int_A \omega_h \in r\mathbb{Z}$ for any closed cycle A . The scalar component ω_h is a scalar free field with covariance kernel gG_{Σ} (and is defined modulo additive constant).

Finally, let us mention that, similarly to Kramers-Wannier duality for the Ising model, the free field possesses an abelian duality which is a simple example of *T-duality* ([18, 13]). For a well-chosen normalisation of the field, *T-duality* inverts the compactification radius and exchanges electric (order) operators with magnetic (disorder) operators.

2.2 Dimers

For simplicity, we discuss here dimers on the square lattice \mathbb{Z}^2 ; however we shall subsequently consider the more general framework of isoradial graphs. For background on dimers, see [29] and references within.

We will consider the square lattice itself, or some portion of it, or its quotient by a large scale lattice (yielding a graph embedded on a torus). A *perfect matching* or *dimer configuration* consists in a selection of edges such that each vertex abuts exactly one selected edge. For a finite graph, one may consider the uniform measure on such matchings. A Gibbs measure on matchings of the infinite volume graph \mathbb{Z}^2 may be obtained as the weak limit of finite volume measures ([10, 31]). The analysis of dimers, in particular in the fine mesh limit, relies extensively on Kasteleyn's Pfaffian enumeration of dimer configurations ([23]) for planar graphs.

Following Thurston, to every dimer configuration on such a graph derived from \mathbb{Z}^2 , one may associate an integer-valued (in some normalisation) height function h on its dual graph. Kenyon established in several set-ups that in the small mesh limit this height function converges to a free field ([27]); this has been extended in [12, 31]. Specifically (for the square lattice), one can obtain a local central limit theorem, showing convergence of discrete current correlators to their free field limit. By integration against a test function, this gives the correct limit in law for the discrete height field, in some finite dimensional marginal sense.

While the relationship between dimer height functions and the *scalar* free field was highlighted by these works, the relation with the *compactified* free field becomes apparent when considering toroidal graphs. There, only the height current is well-defined, and the height function is additively multivalued (when tracing it along non-contractible cycles on the torus): as explained earlier, it splits into an instanton and a scalar component. In the case of the hexagonal lattice, the limiting distribution of the instanton marginal was worked out in [5], in agreement with what is predicted by the compactified free field. This will be extended in the present article.

By analogy with vertex correlators for the free field, one may consider the asymptotics of discrete observables of the type: $\langle \exp(i \sum_j \alpha_j h(z_j)) \rangle$, where ϕ denotes the height function and the weights α_j sum up to zero. The systematic study of these electric vertex correlators for dimer height functions has been initiated by Pinson ([37]) and will be further discussed here.

A classical question for dimers, introduced by Fisher and Stephenson in [17], is the problem of *monomer correlations*. Specifically, the question is to estimate the variation of the partition function when a certain number of vertices are removed from the graph. A motivation for this question is a close analogy with Ising correlations, as illustrated by Hartwig's results ([22]) for monomer pair correlations on the diagonal. More recently, a series of articles by Ciucu (see [9] and references therein) has born on several variants of monomer correlations, in particular correlations of appropriate “islets” on the hexagonal lattice. In terms of height function, monomer insertions may be thought of as magnetic charges. Indeed, in the presence of monomers, the height function becomes additively multivalued, picking fixed additive constants when cycling around monomers (or larger defects).

2.3 Families of Cauchy-Riemann operators

We shall be mostly concerned with establishing convergence of dimer observables (height field, vertex correlators) to the corresponding (compactified) free field quantities. In all cases, the arguments will be based on the analysis of families of Cauchy-Riemann operators and their discrete counterparts.

A Cauchy-Riemann operator in the sense of Quillen ([38]) is an operator $D : \Omega^0(L) \rightarrow \Omega^{0,1}(L)$ of type: $D = d\bar{z}(d_{\bar{z}} + \alpha(z))$ operating on sections of a complex line bundle L over a compact Riemann surface Σ , say. These operators constitute an affine (infinite dimensional) complex space. Given a metric on Σ, L , one can define an adjoint operator D^* and the Laplacian-type operator D^*D . The latter has a ζ -regularised determinant $\det_{\zeta}(D^*D)$ (eg [4]). In [38], it is shown in particular how to evaluate the logarithmic variation of this determinant when D varies among Cauchy-Riemann operators. This is controlled by the asymptotic expansion of the inverting kernel of D near the diagonal. More precisely, if D is invertible, we have the near diagonal expansion (in the standard local coordinate):

$$D^{-1}(z, w) = D_w^{-1}(z, w) + r_{\alpha}(w) + O(|z - w|)$$

where D_w^{-1} is the parametrix: $D_w^{-1}(u, v) = \frac{1}{\pi(u-v)} \exp(2i\Im(a(w)\overline{(u-v)}))$ and $\alpha(w) = a(w)d\bar{w}$ (D_w^{-1} inverts the translation invariant operator that agrees with D at w). The curvature formula in [38] is based on the variational identity (specialised here to the simple case of line bundles on a flat torus):

$$\frac{d}{dt} \log \det_{\zeta}(D_t^* D_t) = \int_{\Sigma} r_{\alpha_t}(w) dw \wedge \dot{\alpha}_t$$

where $D_t = d\bar{z}(\bar{\partial} + \alpha_t(z))$ is a smooth parametric family of CR operators. This is itself a specialisation of the general variational formula for ζ determinants of Laplacian-type operators ([4]).

For instance, consider a torus $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. If we write a section s of a unitary line bundle as $s(z) = f(z) \exp(2i\Re(z\bar{\lambda}))$, the $\bar{\partial}$ operator on this line bundle can be identified with the Cauchy-Riemann operator $D = \bar{\partial} + \lambda d\bar{z}$, and λ is a coordinate on the Jacobian of Σ . In this case, $\det_{\zeta}(D^*D)$ is the (square of) the Ray-Singer *analytic torsion* ([39]). Applying the Quillen variational formula in this case leads to Fay's evaluation of the Second Kronecker Limit formula (up to multiplicative constant). One may also consider CR operators written as $D = \bar{\partial} + g_{\bar{z}} d\bar{z}$; this corresponds to changing the metric on the trivial line bundle (ie scaling the fiber at z by $e^{g(z)}$). In general, we are looking at $\bar{\partial}$ operating on a complex Hermitian line bundle and the variation of the associated ζ -determinant under a change of complex or Hermitian structure.

In order to deal with vertex correlators, we will need to control more singular situations. For instance in the plane with n punctures $\Sigma = \mathbb{C} \setminus \{z_1, \dots, z_n\}$, consider a fixed unitary character χ of $\pi_1(\Sigma)$ and the associated complex line bundle $L_{(z_j)}^{\chi}$ over Σ and $\bar{\partial}$ operator. There the punctures induce a logarithmic

divergence in the evaluation of $\det_\zeta(\bar{\partial}_L^* \bar{\partial}_L)$ and an additional principal part regularisation is needed (see [36]). We will consider two types of variations: a moduli space (isomonodromic) variation (moving the punctures and keeping the character fixed) and a Jacobian variation (changing the character/complex structure of the line bundle with fixed punctures).

On the discrete side, Kasteleyn ([23]) has shown how to evaluate the partition function of dimer configurations on a planar graph as the determinant (in the bipartite case) of a nearest neighbour linear operator. This operator may be interpreted as a finite difference version of $\bar{\partial}$, which is reflected in the asymptotic expansion of its inverse ([26, 28, 31]) and underpins the analysis of the scaling limit of dimers.

We will be relying systematically on this interpretation: in order to evaluate variations of the partition function, we will consider the associated modified Kasteleyn operator as a discrete version of a CR operator varying in a family. Variational formulae, both at the discrete and continuous level, involve the behaviour of the inverting kernel near the diagonal (after subtracting the leading singularity, which is that of a translation-invariant operator). Convergence of observables will follow from convergence of these leading non-trivial coefficients in the short-range expansion of inverting kernels. For vertex correlators, this will involve a rather detailed study of discrete holomorphic functions with prescribed monodromy.

The correspondence between observables and families of CR operators (or Hermitian line bundles) goes as follows:

- instanton component \leftrightarrow variation of the complex structure of the line bundle
- scalar field \leftrightarrow variation of the Hermitian norm of the line bundle
- variation of electric vertex correlators $\langle : \exp(i \sum_j \alpha_j \phi(z_j)) : \rangle$ wrt position of insertions \leftrightarrow family $(z_1, \dots, z_n) \mapsto L_{(z_j)}^\chi$
- variation of electric vertex correlators $\langle : \exp(i \sum_j \alpha_j \phi(z_j)) : \rangle$ wrt charges \leftrightarrow family $\chi \mapsto L_{(z_j)}^\chi$

Finally, for monomer correlations, for instance in the simplest case of pair correlations between points x and y , the relevant line bundle over $\hat{\mathbb{C}} \setminus \{x, y\}$ has local sections s with monodromy (-1) around x, y and $s(z) = O(\sqrt{\frac{z-x}{z-y}})$ near x, y . The variational analysis then consists in displacing one the punctures.

In recent work ([30]), Kenyon employs related variational argument for rank 2 bundles in his study of the double dimer model.

3 Dimers on isoradial graphs

In this section, we are following the formalism of [35, 28, 7] for isoradial (critical) graphs. Notations and conventions are mostly as in [7].

3.1 Kasteleyn operator

Consider a tiling Λ of the plane by rhombi with edge length δ . As faces have even degree, it is bipartite (ie there is a 2-colouring of vertices, say red and blue). Thus one may obtain two graphs from this tiling: the vertices of Γ are the blue vertices and are joined by an edge if they are vertices of the same rhombus; similarly the vertices Γ^\dagger are the red vertices of Λ . The graphs Γ and Γ^\dagger are dual. They are also *isoradial* in the sense that each face of Γ is inscribed in a circle of radius δ , the center of which is the corresponding vertex of the dual Γ^\dagger . The dual of Λ is denoted \diamond . By abuse of terminology, we sometimes identify a graph with the set of its vertices.

We will work under the following assumption (see [7]):

(♠) : the lozenge angles belong to $[\eta_0, \pi - \eta_0]$ for some fixed $\eta_0 \in (0, \pi)$

Here $\eta_0 > 0$ is fixed once and for all; throughout, “absolute” constants may depend on η_0 (but not on Λ etc).

One can form a new bipartite graph M as follows: black (resp. white) vertices of M are the vertices (resp. centers) of rhombi in Λ . Edges of M are half-diagonals of the rhombi in Λ . We will be interested in perfect matchings of M (or subgraphs of M). We refer to vertices of M as *nodes*. Black nodes corresponding to vertices of Γ (resp. vertices of Γ^\dagger) are *vertex nodes* (resp. *face nodes*); white nodes are *edge nodes*. Denote $M_V \simeq \Gamma$, $M_F \simeq \Gamma^\dagger$, $M_W \simeq \diamond$ the sets of vertex nodes, face nodes and edge nodes (ie white nodes) respectively, and $M_B = M_V \sqcup M_F \simeq \Lambda$ (black nodes). See Figure 1.

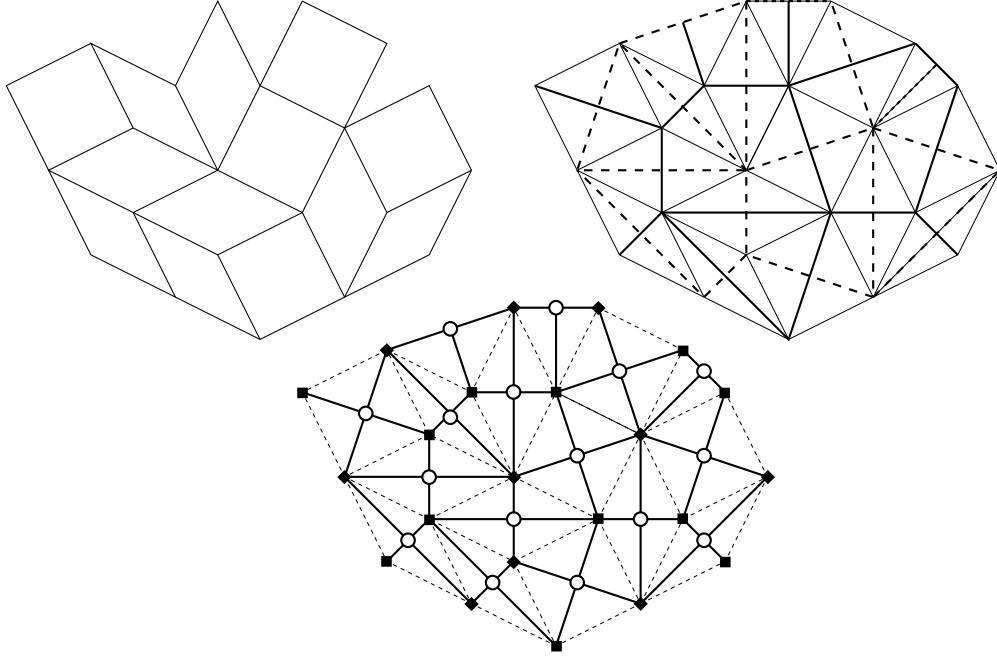


Figure 1: (1) A portion of a rhombus tiling Λ . (2) Dual graphs Γ (solid) and Γ^\dagger (dashed) obtained from Λ . (3) Corresponding bipartite graph M .

Let M_B (resp. M_W) be the set of black (resp. white) vertices of M . The *Kasteleyn operator* $K : \mathbb{C}^{M_B} \rightarrow \mathbb{C}^{M_W}$ is given by ([28]):

$$(Kf)(w) = \sum_{b \sim w} K(w, b) f(b)$$

where $K(w, b_0) = \frac{i}{2}(b_3 - b_1)$ if b_0, b_1, b_2, b_3 are the black neighbours of w listed in counterclockwise order.

For our purposes, it will be convenient to consider a real operator $\mathsf{K} : \mathbb{R}^{M_B} \rightarrow \mathbb{R}^{M_W}$ which differs from K by the argument of its entries: $\mathsf{K}(w, b) = \pm |K(w, b)|$, where the sign depends of a chosen orientation of $(w, b) \in E_M$, which now we describe (see [11]). Pick an arbitrary orientation of edges of Γ ; then orient edges of Γ^\dagger in such a way that if (xx') is an oriented edge of Γ , (yy') the dual oriented edge of Γ^\dagger , then $((xx'), (yy'))$ is a direct frame. Each edge of M is a half-edge of an edge of either Γ or Γ^\dagger and inherits its orientation. Then $\mathsf{K}(w, b) = +|K(w, b)|$ (resp. $\mathsf{K}(w, b) = -|K(w, b)|$) if (wb) is positively (resp. negatively) oriented. One readily checks that this yields a *Kasteleyn orientation* of M , i.e. an orientation such that each face has an odd number of clockwise oriented edges on its boundary.

If $w \in M_W$ is on the oriented edge (xx') of Γ , set $e^{i\nu(w)} = \frac{x' - x}{|x' - x|}$; also set $e^{i\nu(b)} = 1$ if $b \in \Gamma$ is a vertex node and $e^{i\nu(b)} = i$ if $b \in \Gamma^\dagger$ is a face node. Then it is easy to check that

$$K(w, b) = e^{i\nu(w)} \mathbf{K}(w, b) e^{i\nu(b)}$$

for all b, w , viz. \mathbf{K} is obtained by composing K with diagonal operators.

Translating results of [28] obtained for K in terms of \mathbf{K} (see also [7]), we have:

Theorem 1 ([28]). 1. *There is a unique kernel $\underline{\mathbf{K}}^{-1} : \mathbb{R}^{M_B} \times \mathbb{R}^{M_W} \rightarrow \mathbb{R}$ such that for all $w \in M_W$, $\mathbf{K}\underline{\mathbf{K}}^{-1}(., w) = \delta_w$ and $\underline{\mathbf{K}}^{-1}(b, w) = o(1)$ as $|b - w| \rightarrow \infty$.*

2. *If x, y, x', y' are black vertices in ccwise order around w ,*

$$\mathbf{K}(w, x) \underline{\mathbf{K}}^{-1}(x, w) = \frac{1}{2\pi} \arg \left(\frac{y' - x}{y - x} \right).$$

3. *As $|b - w| \rightarrow \infty$,*

$$\underline{\mathbf{K}}^{-1}(b, w) = \Re \left(\frac{e^{i\nu(w)} e^{i\nu(b)}}{\pi(b - w)} \right) + O \left(\frac{\delta}{|b - w|^2} \right)$$

Proof. See respectively Theorem 4.1, Theorem 4.2 and Corollary 7.4 in [28], and also Theorem 2.14 in [7] for the general case. \square

On \mathbb{R}^{M_B} , take the inner product given by counting measure and on \mathbb{R}^{M_W} :

$$\langle f, g \rangle_{\mathbb{R}^{M_W}} = \sum_w \mu_\diamond(w) f(w) g(w)$$

where $\mu_\diamond(w)$ is the area of the rhombus of Λ containing w . With respect to these inner products, the adjoint operator $\mathbf{K}^* : \mathbb{R}^{M_W} \rightarrow \mathbb{R}^{M_B}$ has matrix elements:

$$\mathbf{K}^*(b, w) = \mu_\diamond(w)^{-1} \overline{\mathbf{K}}(w, b)$$

Then it is easy to check that $(\mathbf{K}^* \mathbf{K})(b, b') = 0$ for all $b \in \Gamma, b' \in \Gamma^\dagger$, and if $x \sim x'$ in Γ and (yy') is the edge of Γ^\dagger dual to (xx') , then

$$\begin{aligned} (\mathbf{K}^* \mathbf{K})(x, x') &= \frac{|y' - y|}{2|x' - x|} & (\mathbf{K}^* \mathbf{K})(x, x) &= - \sum_{x' \sim x} (\mathbf{K}^* \mathbf{K})(x, x') \\ (\mathbf{K}^* \mathbf{K})(y, y') &= \frac{|x' - x|}{2|y' - y|} & (\mathbf{K}^* \mathbf{K})(y, y) &= - \sum_{y' \sim y} (\mathbf{K}^* \mathbf{K})(y, y') \end{aligned}$$

so that with respect to the decomposition $\mathbb{R}^{M_B} = \mathbb{R}^{M_V} \oplus \mathbb{R}^{M_F}$, $\mathbf{K}^* \mathbf{K}$ splits as $\Delta_\Gamma \oplus \Delta_{\Gamma^\dagger}$, where Δ is a graph Laplacian for appropriate edge weights ([28]).

Define restriction operators $\mathbf{R}_B : C^0(\mathbb{R}^2, \mathbb{C}) \rightarrow \mathbb{R}^{M_B}$ and $\mathbf{R}_W : C^{(0,1)}(\mathbb{R}^2, \mathbb{C}) \rightarrow \mathbb{R}^{M_W}$ by:

$$\begin{aligned} (\mathbf{R}_B \phi)(b) &= \Re(e^{i\nu(b)} \phi(b)) & b &\in M_B \\ (\mathbf{R}_W \alpha)(w) &= \Re(e^{-i\nu(w)} \alpha/d\bar{z}) & w &\in M_W \end{aligned}$$

that restrict complex-valued functions (resp. $(0, 1)$ -forms) to black (resp. white) nodes. Note that these operators are only \mathbb{R} -linear. If ϕ is a C^1 function on \mathbb{C} , a direct computation shows that:

$$\begin{aligned} (\mathbf{K}(\mathbf{R}_B\phi))(w) &= \frac{1}{2}\Re\left(|y-y'|(\phi(w+u.\frac{|x'-x|}{2})-\phi(w-u.\frac{|x'-x|}{2}))+i|x-x'|(\phi(w+iu.\frac{|y'-y|}{2})-\phi(w-iu.\frac{|y'-y|}{2}))\right) \\ &= 2\mu_\diamond(w)(\Re(\partial_{\bar{z}}\phi(w)\bar{u})+o(1))=2\mu_\diamond(w)(\mathbf{R}_W(\bar{\partial}\phi)(w)+o(1)) \end{aligned}$$

where (xx') is the oriented edge of Γ corresponding to w , (yy') the dual edge, $u = e^{i\nu(w)}$ and $\mu_\diamond(w)$ is the area of the rhombus of Λ containing w (and thus of order $O(\delta^2)$).

Let x_1, \dots, x_n be the neighbours of a vertex x in Γ , in ccwise order (with cyclical indexing $x_n = x_0$). Let y_k be the vertex of Γ^\dagger corresponding to the face of Γ on the left-hand side of the oriented edge (xx_k) and $w_k \in M_W$ the vertex corresponding to (xx_k) . Then for $\alpha = \lambda d\bar{z}$, $\lambda \in \mathbb{C}$ constant,

$$\begin{aligned} \sum_{k=1}^n \mu_\diamond(w_k) e^{i\nu(w_k)} (\mathbf{R}_W\alpha)(w_k) &= \sum_{k=1}^n \frac{i}{2} (y_{k-1} - y_k) \Re(\lambda \overline{(x_k - x)}) \\ &= \sum_{k=1}^n \frac{i}{4} (\lambda (y_{k-1} - y_k) \overline{(x_k - x)} + \bar{\lambda} (y_{k-1} - y_k) (x_k - x)) = \frac{\lambda}{2} \sum_{k=1}^n \mu_\diamond(w_k) \end{aligned}$$

taking into account $x_k - x = (y_{k-1} - x) + (y_k - x)$ and $\mu_\diamond(w_k) = \frac{i}{2} (y_{k-1} - y_k) \overline{(x_k - x)}$. Hence if ϕ is continuous on a region $D \subset \mathbb{C}$,

$$\sum_{w \in \delta M_B \cap D} \mu_\diamond(w) e^{i\nu(w_k)} (\mathbf{R}_W\phi)(w) = \frac{1}{2} \int_D \frac{\alpha}{d\bar{z}} dA + o(1) = -\frac{i}{4} \int_D \alpha \wedge dz + o(1) \quad (3.1)$$

since $dA = dx \wedge dy = -\frac{i}{2} dz \wedge d\bar{z}$. Similarly,

$$\begin{aligned} \sum_{k=1}^n \Delta_\Gamma(x, x_k) (x_k - x) &= \sum_{k=1}^n \frac{i(y_{k-1} - y_k)}{2(x_k - x)} (x_k - x) = 0 \\ \sum_{k=1}^n \Delta_\Gamma(x, x_k) (x_k - x)^2 &= \sum_{k=1}^n \frac{i(y_{k-1} - y_k)}{2(x_k - x)} (x_k - x)^2 = 0 \end{aligned}$$

which shows that the random walk on Γ associated to Δ_Γ converges to isotropic Brownian motion.

We will also consider families of operators parameterized by a $(0, 1)$ -form $\alpha = a(z)d\bar{z}$:

$$\mathbf{K}_\alpha(w, b) = \mathbf{K}(w, b) \exp\left(2\Re \int_w^b \alpha\right)$$

As before, if ϕ is C^1 , we have

$$\begin{aligned} (\mathbf{K}_\alpha(\mathbf{R}_B\phi))(w) &= \frac{\mu_\diamond(w)}{2} \left(4\mathbf{R}_W(\bar{\partial}\phi) + 4\Re(a\bar{u})\Re(\phi) + 4\Re(a\overline{iu})\Re(i\phi) + o(1)\right) \\ &= 2\mu_\diamond(w) (\mathbf{R}_W(\bar{\partial}\phi + \phi\alpha) + o(1)) \end{aligned}$$

where $\phi = \phi(w)$, $a = a(w)$, $u = e^{i\nu(w)}$.

In the case $\alpha = \lambda d\bar{z}$ for some fixed $\lambda \in \mathbb{C}$, we denote simply $\mathbf{K}_\lambda = \mathbf{K}_{\lambda d\bar{z}}$. We have $\mathbf{K}_\lambda(w, b) = e^{-2\Re(\bar{\lambda}w)} \mathbf{K}(w, b) e^{2\Re(\bar{\lambda}b)}$ and consequently

$$\underline{\mathbf{K}}_\lambda^{-1}(b, w) \stackrel{def}{=} e^{-2\Re(\bar{\lambda}b)} \underline{\mathbf{K}}^{-1}(b, w) e^{2\Re(\bar{\lambda}w)}$$

satisfies $\mathbf{K}_\lambda \underline{\mathbf{K}}_\lambda^{-1} = \text{Id}$.

3.2 Height function

Consider a finite bipartite graph Ξ which is a subgraph of M bounded by a simple closed cycle on M . A *perfect matching* \mathbf{m} of the bipartite graph Ξ is a subset of edges of Ξ such that every vertex of Ξ is incident to exactly one edge in \mathbf{m} . The weight of a matching \mathbf{m} is the product of the weights of the edges present in \mathbf{m} :

$$w(\mathbf{m}) = \prod_{(bw) \in \mathbf{m}} |\mathbf{K}(b, w)|$$

The *partition function* of the model is the sum of these weights over all possible perfect matchings of M :

$$\mathcal{Z} = \sum_{\mathbf{m}} \prod_{(bw) \in \mathbf{m}} |\mathbf{K}(b, w)|$$

We will consider the probability measure on perfect matchings \mathbf{m} of Ξ (if they exist) given by $\mathbb{P}(\mathbf{m} = \mathbf{m}_0) = w(\mathbf{m}_0)/\mathcal{Z}^{-1}$.

In order to have at least one matching, it is necessary that $\Xi_B = \Xi \cap M_B$ and $\Xi_W = \Xi \cap M_W$ have the same number of vertices. Let us also denote by $\mathbf{K} : \mathbb{R}^{\Xi_B} \rightarrow \mathbb{R}^{\Xi_W}$ the restriction of $\mathbf{K} : \mathbb{R}^{M_B} \rightarrow \mathbb{R}^{M_W}$, ie $(\mathbf{K}f)(w) = \sum_{b \in \Xi_B} \mathbf{K}(b, w)f(b)$.

The fundamental result, due to Kasteleyn ([23]), is the following determinantal enumeration formula:

$$\mathcal{Z} = \pm \det(\mathbf{K})$$

(this is where the Kasteleyn orientation condition is required). Notice that $\mathbf{K} : \mathbb{R}^{\Xi_B} \rightarrow \mathbb{R}^{\Xi_W}$ is not an endomorphism; however, \mathbb{R}^{Ξ_B} and \mathbb{R}^{Ξ_W} have canonical bases (up to permutation) with respect to which this determinant is evaluated (up to sign). Also, $\det(K) = \det(\mathbf{K}) \prod_{w \in \Xi_W} e^{i\nu(w)} \prod_{b \in \Xi_B} e^{i\nu(b)}$, so that $\mathcal{Z} = |\det(K)|$.

Moreover, we can also evaluate the partition function for arbitrary complex edge weights. If we replace the positive weight $|\mathbf{K}(b, w)|$ with $|\mathbf{K}(b, w)|u(b, w)$, $u(b, w) \in \mathbb{C}$, then:

$$\mathcal{Z}' = \sum_{\mathbf{m}} \prod_{(bw) \in \mathbf{m}} |\mathbf{K}(b, w)|u(b, w) = \pm \det(\mathbf{K}')$$

where $\mathbf{K}' : \mathbb{C}^{\Xi_B} \rightarrow \mathbb{C}^{\Xi_W}$ is given by $(\mathbf{K}'f)(w) = \sum_{b \in \Xi_B} \mathbf{K}(b, w)u(b, w)f(b)$. The undetermined sign is the same as before, and thus:

$$\mathbb{E}\left(\prod_{(bw) \in \mathbf{m}} u(b, w)\right) = \frac{\mathcal{Z}'}{\mathcal{Z}} = \frac{\det \mathbf{K}'}{\det \mathbf{K}} = \det(\mathbf{K}'\mathbf{K}^{-1}) = \det(K'K^{-1})$$

where $(K'f)(w) = \sum_{b \in \Xi_B} K(b, w)u(b, w)f(b)$

Let us now discuss the height function, following [33, 29]. The height function is defined on the “dual” Ξ^\dagger , defined here as the subgraph of M^\dagger whose vertices correspond to faces of M that are adjacent to vertices of Ξ ; we can take the vertices of Ξ^\dagger to be the midpoints of edges of Λ . Given a perfect matching \mathbf{m} of the bipartite graph Ξ , one can define a closed 1-form on its dual Ξ^\dagger (ie. an antisymmetric function on oriented edges of Ξ^\dagger) as follows. Consider the 1-form $\omega \in C^1(\Xi^\dagger)$ defined by:

$$\omega((bw)^\dagger) = \begin{cases} 1 & \text{if } (bw) \text{ matched, oriented from black to white} \\ 0 & \text{otherwise} \end{cases}$$

Then $d\omega \in C^2(\Xi^\dagger) \sim C^0(\Xi)$ is 1 on black vertices, -1 on white vertices, where $(d\omega)(f)$ is the sum of ω on the edges bounding the face f of Ξ^\dagger , oriented counterclockwise. Given the embedding of Ξ in the plane, one

can construct a fixed 1-form $\omega_0 \in C^1(\Xi^\dagger)$ as follows (here (bw) is an edge of Ξ ; the black neighbours of w are x, y, x', y' in this order):

$$\omega_0((bw)^\dagger) = \frac{1}{2\pi} \arg \frac{y' - x}{y - x} = \mathbf{K}(w, b) \underline{\mathbf{K}}^{-1}(b, w) \stackrel{def}{=} p(w, b)$$

by Theorem 1. Then $d\omega_0 = d\omega$ for any ω defined from a perfect matching as above. This is clear from the local geometry.

Since $d(\omega_0 - \omega) = 0$, we can write

$$\omega_0 - \omega = dh$$

for some function h on Ξ^\dagger , which is uniquely defined up to an additive constant. In the case of graphs in multiply connected domains and on a torus, h is additively multivalued (when tracing h along a non-contractible cycle).

We now seek an interpretation of perturbed operators in terms of height functions. Define

$$\begin{aligned} K_\alpha(w, b) &= \mathbf{K}(w, b) \exp(2i \Im \int_w^b \alpha) \\ \mathbf{K}_\alpha(w, b) &= \mathbf{K}(w, b) \exp(2\Re \int_w^b \alpha) \end{aligned}$$

where α is a smooth $(0, 1)$ -form: $\alpha = a(z)d\bar{z}$.

As earlier we get

$$\begin{aligned} \mathbb{E} \left(\exp(2 \sum_{(bw) \in \mathfrak{m}} \Re \int_w^b \alpha) \right) &= \det(\mathbf{K}_\alpha \mathbf{K}^{-1}) \\ \mathbb{E} \left(\exp(2i \sum_{(bw) \in \mathfrak{m}} \Im \int_w^b \alpha) \right) &= \det(K_\alpha \mathbf{K}^{-1}) \end{aligned}$$

In the simply-connected case (ie when h is single valued), one can write:

$$\mathbf{1}_{(bw) \in \mathfrak{m}} = \omega((ff')) = h(f) - h(f') + \omega_0((ff'))$$

where $(ff') = (bw)^\dagger$. Then

$$\sum_{(bw) \in \mathfrak{m}} \Re \int_w^b \alpha = \sum_{f \in \Xi^\dagger} h(f) (\Re \int_{\partial f} \alpha) + P(\alpha)$$

where ∂f is the boundary of the face f (taken counterclockwise) and $P(\alpha)$ is the \mathbb{R} -linear form:

$$P(\alpha) = \sum_{(bw) \in E_\Xi} p(w, b) \Re \int_w^b \alpha$$

Let us extend h (initially defined on Ξ^\dagger) to a piecewise constant function, ie constant in each face of Ξ . Then by Stokes' formula

$$\sum_{f \in \Xi^\dagger} h(f) (\Re \int_{\partial f} \alpha) = \Re \int_\Xi h \partial \alpha$$

(here Ξ designates the domain of \mathbb{C} covered by the graph; $d\alpha = \partial\alpha$ as α is a $(0,1)$ -form). Finally we obtain the following expression for the Laplace and Fourier functional of the height field:

$$\begin{aligned}\mathbb{E}\left(\exp(2\Re \int_{\Xi} h\partial\alpha)\right) &= \det(\mathbf{K}_{\alpha}\mathbf{K}^{-1}) \exp(-2P(\alpha)) \\ \mathbb{E}\left(\exp(2i\Im \int_{\Xi} h\partial\alpha)\right) &= \det(K_{\alpha}\mathbf{K}^{-1}) \exp(2iP(i\alpha))\end{aligned}$$

The characteristic functional is bounded and is more practical is the absence of boundary. The Laplace functional preserves the real structure ($\mathbf{K}_{\alpha} : \mathbb{R}^{\Xi_B} \rightarrow \mathbb{R}^{\Xi_W}$), which is useful for domains with boundary.

4 Planar graphs

In order to avoid complications related to non-trivial homology or boundaries, we now discuss the case of the plane.

Let $\Lambda = \Lambda_{\delta}$ be a lozenge tiling of the complex plane \mathbb{C} with edge length δ , with δ going to zero along some sequence; Γ, Γ^{\dagger} is the pair of associated isoradial graphs. We assume that condition (\spadesuit) is satisfied for all δ (for a fixed η_0).

From [24, 31, 11], we know that there is a Gibbs measure on perfect matchings of Ξ such that, for any finite subset $\{(b_i, w_i), 1 \leq i \leq n\}$ of edges of Ξ ,

$$\mathbb{P}((b_1, w_1) \in \mathbf{m}, \dots, (b_n, w_n) \in \mathbf{m}) = \left(\prod_{i=1}^n \mathbf{K}(w_i, b_i) \right) \det_{1 \leq i, j \leq n} (\mathbf{K}^{-1}(b_i, w_j))$$

(manifestly, these *local statistics* completely specify the measure). In the case where Λ is the local limit of a sequence of biperiodic lattices, the measure can be realised as the limit of uniform dimer covers of (finite volume) toroidal graphs.

Thus we have a probability measure on matchings of $\Xi = \Xi_{\delta}$; $\mathbb{E} = \mathbb{E}_{\delta}$ is the expectation under this measure, h the height function, seen as a function constant on faces of Ξ and well-defined modulo a global additive constant.

Consider $g \in C_c^1(\mathbb{C})$ a function with compact support, continuous first derivatives and second derivatives in L_{loc}^1 ; set $\alpha = \bar{\partial}g$; we consider again the perturbation

$$K_{\alpha}(w, b) = \mathbf{K}(w, b) \exp(2i\Im \int_w^b \alpha)$$

Since g has compact support, $K_{\alpha}\mathbf{K}^{-1}$ is a finite rank perturbation of the identity, and consequently $\det(K_{\alpha}\mathbf{K}^{-1})$ is well-defined as a Fredholm determinant.

Lemma 2. *Let h be the height function (constant on faces); then*

$$\begin{aligned}\det(K_{\alpha}\mathbf{K}^{-1}) &= \mathbb{E}(\exp(2i\Im \int_{\mathbb{C}} h\partial\bar{\partial}g)) \exp(2i \sum_{w \sim b} p(w, b) \Im \int_w^b \alpha) \\ &= \mathbb{E}(\exp(-i\Re \int_{\mathbb{C}} h(\Delta g) dA)) \exp(-2iP(i\alpha))\end{aligned}$$

Proof. We cannot apply directly the finite volume identity we observed earlier. Let us consider v a finitely supported function on edges of Ξ . Then

$$\prod_{(bw) \in E_{\Xi}} (1 + v(w, b) \mathbf{1}_{(bw) \in \mathbf{m}}) = \sum_{S \subset E_{\Xi}} \prod_{(b, w) \in S} \mathbf{1}_{(bw) \in \mathbf{m}} v(w, b)$$

(with finitely many nonzero summands on the right hand side and by convention $\prod_{\emptyset} = 1$) and thus

$$\mathbb{E} \left(\prod_{(bw) \in E_{\Xi}} (1 + v(w, b) \mathbf{1}_{(bw) \in \mathfrak{m}}) \right) = \sum_{S = \{(b_i, w_i), 1 \leq i \leq n\} \subset E_{\Xi}} \left(\prod_{i=1}^n \mathsf{K}(w_i, b_i) v(w_i, b_i) \right) \det_{1 \leq i, j \leq n} (\mathsf{K}^{-1}(b_i, w_j))$$

On the other hand, if $K' : \mathbb{C}^{\Xi_B} \rightarrow \mathbb{C}^{\Xi_W}$ is given by its matrix elements $K'(w, b) = (1 + v(w, b))\mathsf{K}(w, b)$, the Fredholm expansion (eg [41]) reads:

$$\det(\text{Id} + (K' - K)\mathsf{K}^{-1}) = \sum_{\{w_i, 1 \leq i \leq n\} \subset \Xi_W} \det_{1 \leq i, j \leq n} (((K' - K)\mathsf{K}^{-1})(w_i, w_j))$$

The Cauchy-Binet formula gives

$$\det_{1 \leq i, j \leq n} (((K' - K)\mathsf{K}^{-1})(w_i, w_j)) = \sum_{\{b_k, 1 \leq k \leq n\} \subset \Xi_B} \det_{1 \leq i, k \leq n} ((K' - K)(w_i, b_k)) \det_{1 \leq k, j \leq n} (\mathsf{K}^{-1}(b_k, w_j))$$

and besides

$$\det_{1 \leq i, k \leq n} ((K' - K)(w_i, b_k)) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n \mathsf{K}(w_i, b_{\sigma(i)}) v(w_i, b_{\sigma(i)})$$

This completes the identification (reordering rows of $\det_{1 \leq i, j \leq n} (\mathsf{K}^{-1}(b_i, w_j))$ absorbs $\text{sgn}(\sigma)$):

$$\mathbb{E} \left(\prod_{(bw) \in E_{\Xi}} (1 + v(w, b) \mathbf{1}_{(bw) \in \mathfrak{m}}) \right) = \det(K' \mathsf{K}^{-1})$$

Specialising to $K' = \mathsf{K}_{\alpha}$ concludes (the rest of the argument being as in the finite volume case). \square

The next step is to construct and estimate (in particular near the diagonal) a kernel S_{α} inverting K_{α} .

Let us first consider K_{α} as a finite difference operator. For ϕ a function on \mathbb{C} , we define restriction operators R_B, \bar{R}_B by: $(R_B \phi)(b) = (\bar{R}_B \phi)(b) = \phi(b)$ for $b \in \Gamma$ and $(R_B \phi)(b) = -(\bar{R}_B \phi)(b) = i\phi(b)$ for $b \in \Gamma^{\dagger}$. Let us also denote $(R_W \beta)(w) = b(w)e^{-i\nu(w)}$ for $\beta = b(z)d\bar{z}$ and $(\bar{R}_W \beta) = b(w)e^{i\nu(w)}$ for $\beta = b(z)dz$. These restriction operators are \mathbb{C} -linear.

Let $w \in M_W$ be an edge node, with black neighbours x, y, x', y' in ccwise order labelled in such a way that (xx') is an oriented edge of Γ . We have

$$\mathsf{K}(R_B \phi) = 2\mu_{\diamond} R_W(\bar{\partial}\phi) + O(\delta^2 \omega_{\phi'}(\delta)) \quad (4.2)$$

$$= 2\mu_{\diamond} R_W(\bar{\partial}\phi) + O(\delta^3 \omega_{\phi''}(\delta)) \quad (4.3)$$

where ω designates the modulus of continuity, ϕ' and ϕ'' the gradient and Hessian of ϕ , and μ_{\diamond} is seen as a diagonal operator on \mathbb{C}^{M_W} .

Observe that

$$K_{\alpha}(R_B \phi)(w) = \mathsf{K}(R_B(\tilde{\phi}))(w)$$

where $\tilde{\phi}(z) = \phi(z) \exp(2i\Im \int_w^z \alpha)$. We deduce (here $a = g_{\bar{z}}$)

$$K_{\alpha}(R_B \phi) = 2\mu_{\diamond} R_W(\bar{\partial}\phi + \phi\alpha) + \delta^2 O(\omega_{\phi'}(\delta) + \delta \|\phi'\|_{\infty} \|a\|_{\infty} + \|\phi\|_{\infty} \omega_a(\delta) + \|\phi\|_{\infty} \delta \|a\|_{\infty}^2) \quad (4.4)$$

$$= 2\mu_{\diamond} R_W(\bar{\partial}\phi + \phi\alpha) + O(\delta^3 \|\phi\|_{C^2} (1 + \|a\|_{C^1})^2) \quad (4.5)$$

$$= 2\mu_{\diamond} R_W(\bar{\partial}\phi + \phi\alpha) + O(\delta^4 \|\phi\|_{C^3} (1 + \|a\|_{C^2})^3) \quad (4.6)$$

Similarly, we obtain

$$K_\alpha(\bar{R}_B\phi) = 2\mu_\diamond \bar{R}_W(\partial\phi - \phi\bar{\alpha}) + O(\delta^4\|\phi\|_{C^3}(1 + \|a\|_{C^2})^3)$$

Fix $w_0 \in M_W$ and set $\lambda = a(w_0)$. Then:

$$K_\alpha(w, b) = e^{2i\Im(\bar{\lambda}w)}\mathbf{K}(w, b)e^{-2i\Im(\bar{\lambda}b)} + O(\delta^2\omega_a(|b - w_0|)) \quad (4.7)$$

The finite difference lead us to think of K_α as a finite difference version of (simultaneously) $\bar{\partial} + \alpha$ and the adjoint operator $\partial - \bar{\alpha}$. To construct a kernel \mathbf{S}_α inverting K_α , we are going to construct an approximate kernel $\tilde{\mathbf{S}}_\alpha$ using at large scale inverting kernels for these continuous Cauchy-Riemann operators, at small scale ($|b - w| \ll 1$) discrete holomorphic functions; and finally control the error between $\tilde{\mathbf{S}}_\alpha$ and \mathbf{S}_α .

We have $\alpha = \bar{\partial}g$. Thus $\bar{\partial} + \alpha = e^{-g}(\bar{\partial})e^g$, and consequently

$$S_\alpha(z, w) = \frac{e^{g(w)-g(z)}}{\pi(z-w)}$$

is a kernel inverting $(\bar{\partial} + \alpha)$ (on the right): $(\bar{\partial} + \alpha)S_\alpha(\cdot, w) = \delta_w d\bar{z}$ as distributions. It is uniquely characterized by $S_\alpha(z, w) \rightarrow 0$ as $z \rightarrow \infty$.

Set

$$\tilde{\mathbf{S}}_\alpha(b, w) = e^{2i\Im(\bar{\lambda}(b-w))}(\underline{\mathbf{K}}^{-1}(b, w) + R_B(\mu) + \bar{R}_B(\mu'))$$

for $|b - w| < \eta$ and

$$\tilde{\mathbf{S}}_\alpha(b, w) = \frac{1}{2} \left(R_B(e^{i\nu(w)}S_\alpha(b, w)) + \bar{R}_B(e^{-i\nu(w)}\overline{S_{-\alpha}(b, w)}) \right)$$

for $|b - w| \geq \eta$, where η, μ, μ' are parameters to be specified.

We have

$$\begin{aligned} S_\alpha(b, w)e^{2i\Im(\bar{\lambda}(b-w))} &= \frac{1}{\pi(b-w)}e^{2i\Im(\bar{g}_z(w)(b-w)-(g(b)-g(w)))} \\ &= \frac{1}{\pi(b-w)} - \frac{1}{\pi}(\bar{g}_z(w) + g_z(w)) + O(b-w) = \frac{1}{\pi(b-w)} - \frac{2}{\pi}(\partial_z \Re g)(w) + O(e^{2\|g\|_\infty}\omega_{g'}(|b-w|)) \end{aligned}$$

We can estimate $K_\alpha \tilde{\mathbf{S}}_\alpha(\cdot, w)$. If w' is such that all its black neighbours are in $D(w, \eta)$, given that $\underline{\mathbf{K}}^{-1}(b, w) = O(|b - w|^{-1})$, we get from (4.7):

$$(K_\alpha \tilde{\mathbf{S}}_\alpha(\cdot, w))(w') = \delta_w(w') + O\left(\delta^2 \frac{\omega_{g'}(|w' - w| + \delta)}{|w' - w| + \delta}\right)$$

If all black neighbours of w' are outside $D(w, \eta)$, we have as in (4.4)

$$(K_\alpha \tilde{\mathbf{S}}_\alpha(\cdot, w))(w') = \delta^2 e^{2\|g\|_\infty} O\left(\frac{\omega_{g'}(\delta)}{|w' - w|} + \frac{\delta\|g'\|_\infty}{|w' - w|^2} + \frac{\delta}{|w' - w|^3} + \frac{\delta\|g'\|_\infty^2}{|w' - w|}\right)$$

and for w' outside of $D(w, \eta) \cup \text{supp}(g)$,

$$(K_\alpha \tilde{\mathbf{S}}_\alpha(\cdot, w))(w') = e^{\|g\|_\infty} O\left(\frac{\delta^4}{|w' - w|^4}\right)$$

We have seen that

$$\underline{\mathbf{K}}^{-1}(b, w) = \frac{1}{2}R_B\left(\frac{e^{i\nu(w)}}{\pi(b-w)}\right) + \frac{1}{2}\bar{R}_B\left(\frac{e^{-i\nu(w)}}{\pi(b-w)}\right) + O\left(\frac{\delta}{|b-w|^2}\right)$$

Set

$$\begin{aligned}\mu &= \frac{1}{2}e^{i\nu(w)}\left(-\frac{2}{\pi}(\partial_z \Re g)(w)\right) \\ \mu' &= \frac{1}{2}e^{-i\nu(w)}\left(\frac{2}{\pi}(\partial_{\bar{z}} \Re g)(w)\right) = -\bar{\mu}\end{aligned}$$

Then if $\text{dist}(b, w)$ is of order η , the difference between the short and long distance definitions of \tilde{S}_α is of order $O(\omega_{g'}(\eta)e^{2\|g\|_\infty} + \delta/\eta^2)$. Consequently, if w' has neighbours both inside and outside of $D(w, \eta)$, $(K_\alpha \tilde{S}_\alpha(\cdot, w))(w') = O(\delta\omega_{g'}(\eta)e^{2\|g\|_\infty} + \delta^2/\eta^2)$.

Assume now that g is in the Sobolev space $W^{2,p}$ with $p > 2$ and has support in $B(0, r)$. In what follows constants may depend on p, r . By Morrey's inequality (eg [1]), g' is ε -Hölder with Hölder norm less than $c\|g''\|_p$ for $\varepsilon = 1 - 2/p$. It follows that $\omega_{g'}(s) \leq c\|g''\|_p s^\varepsilon$ and $\|g\|_{C^1} \leq c\|g''\|_p$. Set $C = \|g''\|_p$.

Thus, the L^1 norm of $K_\alpha \tilde{S}_\alpha(\cdot, w) - \delta_w$ (wrt counting measure) is less than:

$$c \left(\delta^2 \sum_{k=1}^{\eta/\delta} k C (k\delta)^{\varepsilon-1} + (C e^{cC} \delta \eta^\varepsilon + \frac{\delta^2}{\eta^2}) \frac{\eta}{\delta} + \delta^2 e^{cC} \sum_{k=\eta/\delta}^{r/\delta} k \cdot \left(\frac{C \delta^\varepsilon + C^2 \delta}{(k\delta)} + \frac{\delta C}{(k\delta)^2} + \frac{\delta}{(k\delta)^3} \right) + e^{cC} \sum_{k=r/\delta}^{\infty} k \frac{\delta^4}{(k\delta)^4} \right)$$

hence less than

$$c e^{cC} (\eta^{\varepsilon+1} + \frac{\delta}{\eta} + \delta^\varepsilon + \delta |\log \eta| + \frac{\delta}{\eta} + \delta^2)$$

Here we simply collect errors stemming from: replacing K_α with an operator conjugate to K at short distance; gluing the short and long distance approximation; and using the limiting continuous kernel at long distance.

Setting now $\eta = \delta^\beta$ with $\beta = \frac{1}{\varepsilon+2}$, and $T = \text{Id} - K_\alpha \tilde{S}_\alpha$, we get that

$$\|T\|_{L^1} \leq c e^{cC} (\delta^{(\varepsilon+1)/(\varepsilon+2)} \vee \delta^\varepsilon)$$

where $\|\cdot\|_{L^1}$ is the $L^1 \rightarrow L^1$ operator norm, and consequently $(\text{Id} - T) : L^1 \rightarrow L^1$ is invertible for δ small enough, and we may set:

$$S_\alpha \stackrel{\text{def}}{=} \tilde{S}_\alpha \left(\sum_{k=0}^{\infty} T^k \right)$$

so that $K_\alpha S_\alpha = \text{Id}$.

We now want to estimate $\|\tilde{S}_\alpha T\|_{L^1 \rightarrow L^\infty}$. We simply expand

$$(\tilde{S}_\alpha T)(b, w) = \sum_{w'} \tilde{S}_\alpha(b, w') T(w', w)$$

and as before we split T in a short range and long range part and notice that $\tilde{S}_\alpha(b, w) = O(e^{cC}/\text{dist}(b, w))$ to obtain:

$$\begin{aligned}\|(\tilde{S}_\alpha T)(\cdot, w)\|_\infty &\leq c e^{cC} \left(\delta^2 \sum_{k=0}^{\eta/\delta} k (k\delta)^{\varepsilon-2} + (\delta \eta^\varepsilon + \frac{\delta^2}{\eta^2}) \sum_{i=1}^{c\eta/\delta} \frac{1}{\delta i} + \delta^2 \sum_{k=\eta/\delta}^{r/\delta} k \cdot \left(\frac{\delta^\varepsilon}{(k\delta)^2} + \frac{\delta}{(k\delta)^3} \right) + \sum_{k=1}^{\eta/\delta} \frac{k}{k\delta} \left(\frac{\delta^{2+\varepsilon}}{\eta^2} + \frac{\delta^3}{\eta^3} \right) \right. \\ &\quad \left. + \sum_{k=r/\delta}^{\infty} k \frac{\delta^4}{(k\delta)^5} + \sum_{k=1}^{r/\delta} \frac{k}{k\delta} \delta^4 \right) \\ &\leq c e^{cC} (\eta^\varepsilon + (\eta^\varepsilon + \delta^{1-2\beta}) |\log(\delta)| + (\delta^\varepsilon |\log(\delta)| + \delta^{1-\beta} + \delta^{\varepsilon-\beta} + \delta^{1-2\beta}) + \delta^2) = O(e^{cC} \delta^{\varepsilon'})\end{aligned}$$

The various terms correspond to the possible relative positions of b, w', w . Notice however that the estimate is simpler when $b \sim w$, which is the most useful case.

Since $S_\alpha - \tilde{S}_\alpha = (\tilde{S}_\alpha T)(\text{Id} - T)^{-1}$, $\|(\text{Id} - T)^{-1}\|_{L^1} = O(1)$, $\|\tilde{S}_\alpha T\|_{L^1 \rightarrow L^\infty} = O(e^{cC} \delta^{\varepsilon'})$, we conclude that

$$S_\alpha(b, w) - \tilde{S}_\alpha(b, w) = O(e^{c\|g''\|_p} \delta^{\varepsilon'})$$

uniformly in b, w (for p, r fixed; $\varepsilon' > 0$ depends on $p > 2$). In particular for $b \sim w$ we have

$$S_\alpha(b, w) - e^{2i\Im \bar{\lambda}(b-w)} \underline{K}^{-1}(b, w) = i\Im(e^{i\nu} r_\alpha) + O(e^{c\|g''\|_p} \delta^{\varepsilon'})$$

where $r_\alpha = -\frac{2}{\pi}(\partial_z \Re g)(w)$ and $\nu = \nu(b) + \nu(w)$.

Lemma 3. *The following estimate holds*

$$\log \det(K_\alpha \underline{K}^{-1}) = 2i \sum_{b \sim w} p(w, b) \Im \int_w^b \alpha - \frac{1}{2\pi} \int |\nabla \Re g|^2 + O(\delta^{\varepsilon'})$$

Proof. Observe that $K_\alpha \underline{K}^{-1}$ and $\underline{K} S_\alpha$ are finite rank (hence trace class) perturbations of the identity, and $(K_\alpha \underline{K}^{-1})(\underline{K} S_\alpha) = \text{Id}$ (since \underline{K}^{-1} is also a left inverse of \underline{K}). If α depends smoothly on a parameter t (say $\alpha(t) = t\alpha$), we have the variational formula (eg [20], IV.1)

$$\frac{d}{dt} \log \det(K_\alpha S) = \text{Tr}\left(\left(\frac{d}{dt} K_\alpha\right) S_\alpha\right)$$

as long as $K_\alpha \underline{K}^{-1}$ is invertible, which is at least true for small enough t . From

$$\begin{aligned} \dot{K}_\alpha(w, b) &= \underline{K}(w, b) \exp(2i\Im \int_w^b \alpha) 2i\Im \int_w^b \dot{\alpha} \\ S_\alpha(b, w) &= \exp(-2i\Im \int_w^b \alpha) \underline{K}^{-1}(b, w) + i\Im(e^{i(\nu(b)+\nu(w))} r_\alpha) + O(\delta^{\varepsilon'}) \end{aligned}$$

(for $b \sim w$), we obtain (if x, y, x', y' denote the black neighbours of w in ccwise order)

$$\begin{aligned} \dot{K}_\alpha S_\alpha(w, w) &= 2i \sum_{b \sim w} p(w, b) \Im \int_w^b \dot{\alpha} - |y' - y| \cdot |x' - x| \left(\Im(\dot{\lambda} e^{-i\nu(w)}) \Im(e^{i\nu(w)} r_\alpha) + \Im(\dot{\lambda} e^{-i\nu(w)} (-i)) \Im(e^{i\nu(w)} i r_\alpha) \right) + O(\delta^{2+\varepsilon'}) \\ &= 2i \sum_{b \sim w} p(w, b) \Im \int_w^b \dot{\alpha} + 2\mu_\diamond(w) \Re(\dot{\lambda} r_\alpha) + O(\delta^{2+\varepsilon'}) \end{aligned}$$

and

$$\frac{d}{dt} \log \det(K_\alpha \underline{K}^{-1}) = 2i \sum_{b \sim w} p(w, b) \Im \int_w^b \dot{\alpha} + \Im \int \dot{\alpha} \wedge (r_\alpha dz) + O(\delta^{\varepsilon'})$$

(since $dz \wedge d\bar{z} = -2idA$). Moreover, with $\alpha = \bar{\partial}g$,

$$\Im \int \dot{\alpha} \wedge (r_\alpha dz) = -\frac{2}{\pi} \Im \int \bar{\partial} \dot{g} \wedge \partial(\Re g) = \frac{2}{\pi} \Im \int \dot{g} \bar{\partial} \partial(\Re g) = \frac{1}{\pi} \Re \int \dot{g} \Delta g dA = -\frac{1}{2\pi} \frac{d}{dt} \int |\nabla \Re g|^2$$

In particular, for δ small enough $\det(K_\alpha \underline{K}^{-1})$ does not vanish along the interpolation path, thus (eg [41] chapter 3) $K_\alpha \underline{K}^{-1}$ stays invertible and the variational formula is legitimate. \square

From the previous two lemmas we immediately conclude:

Corollary 4. *If $g \in W^{2,p} \cap C_c^0$, $p > 2$, then $\int h \Delta g$ converges in distribution to a centered normal variable with variance $\frac{1}{\pi} \int |\nabla g|^2$, as $\delta \searrow 0$.*

Thus we can integrate the discrete height field h against a test function in L^p , $p > 2$ (the compact support assumption is for technical convenience). An optimal statement (given the free field limit) would involve a test function in H^{-1} ; however h (as defined here) is not in H^1 , due to jump discontinuities on edges of Ξ . It is unclear whether one can define a better interpolation of the discrete function for which one could relax substantially the condition $g \in W^{2,p}$ (to $g \in H^{1+\varepsilon}$, say).

5 Toroidal graphs

As before, we start from a rhombus tiling $\Lambda = \Lambda_\delta$ of the plane with edge length $\delta \ll 1$, from which we construct Γ , Γ^\dagger , $\diamond = \Lambda^\dagger$, M . Additionally, we assume that these structures are bi-periodic, with periods 1 and $\tau = \tau_\delta$ ($\Im \tau > 0$). Denote by Υ the lattice $\mathbb{Z} + \tau\mathbb{Z}$; by quotienting, we obtain a rhombus tiling of $\Sigma = \mathbb{C}/\Upsilon$. We are interested in perfect matchings of $\Xi = M/\Upsilon$. Notice that Euler's formula applied to Γ shows that $|\Xi_B| = |\Xi_W|$.

The height “function” h is associated to a perfect matching \mathbf{m} on Ξ by a local rule. Since the torus is non contractible, the height function is now additively multivalued. The current $J = dh$ is a well-defined closed 1-form on Σ (for some extension of the discrete height function on Ξ^\dagger to a piecewise continuous function on Σ ; then J is a distributional 1-form). The Hodge decomposition of J reads:

$$J = \omega_h + dh_0$$

where ω_h is a closed harmonic form (hence, in flat metric, $\omega_h = adx + bdy$ for some $a, b \in \mathbb{R}$) and h_0 is a (single-valued) function on Σ , well-defined up to an additive constant. The “topological” component ω_0 is uniquely specified by its periods, which equal those of the current: $\int_\gamma \omega_h = \int_\gamma J$, where γ is a basic cycle on Σ .

We are interested in the asymptotic distribution of J , or equivalently the asymptotic joint distribution of (ω_h, h_0) . For clarity we will be discussing first the topological component of the current.

5.1 Flat line bundles

For $\lambda \in \mathbb{C}$, denote:

$$K_\lambda(w, b) = \mathcal{K}(w, b) \exp(2i\Im \int_w^b \lambda d\bar{z}) = \mathcal{K}(w, b) \exp(2i\Im \lambda \overline{(b - w)})$$

so that K_λ defines an operator $\mathbb{C}^{M_B} \rightarrow \mathbb{C}^{M_W}$ and by quotienting an operator $\mathbb{C}^{\Xi_B} \rightarrow \mathbb{C}^{\Xi_W}$. We are concerned with the inverse of that last operator. Let $\chi : \Upsilon \rightarrow \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ be a unitary character; consider the finite dimensional space

$$(\mathbb{C}^{M_B})_\chi = \{f \in \mathbb{C}^{M_B} : \forall z \in M_B, \omega \in \Upsilon, f(z + \omega) = \chi(\omega)f(z)\} \subset \mathbb{C}^{M_B}$$

and $(\mathbb{C}^{M_W})_\chi$ is defined similarly; note that $(\mathbb{C}^M)_{\text{Id}} \simeq \mathbb{C}^\Xi$ (here Id denotes the trivial character). Plainly, as K_λ commutes with the action of Υ by translation, it defines an operator $K_\lambda : (\mathbb{C}^{M_B})_\chi \rightarrow (\mathbb{C}^{M_W})_\chi$. Set :

$$(T_\lambda f)(z) = f(z)e^{2i\Im \lambda \bar{z}}$$

If $\chi_\lambda(\omega) = e^{2i\Im \lambda \bar{\omega}}$ is the character associated to λ , T_λ maps $(\mathbb{C}^M)_{\text{Id}}$ to $(\mathbb{C}^M)_{\chi_\lambda}$, and we have the diagram

$$\begin{array}{ccc} (\mathbb{C}^{M_B})_{\chi_\lambda} & \xrightarrow{\mathcal{K}} & (\mathbb{C}^{M_W})_{\chi_\lambda} \\ \downarrow T_{-\lambda} & & \downarrow T_{-\lambda} \\ (\mathbb{C}^{M_B})_{\text{Id}} & \xrightarrow{K_\lambda} & (\mathbb{C}^{M_W})_{\text{Id}} \end{array}$$

so that we may focus on $\mathcal{K} : (\mathbb{C}^{M_B})_\chi \rightarrow (\mathbb{C}^{M_W})_\chi$. An element in the kernel restricts to a bounded harmonic function on Γ and Γ^\dagger ; then (Liouville) these restrictions are constant, hence the kernel is trivial iff $\chi \neq \text{Id}$.

As $|\Xi_B| = |\Xi_W|$, $K : (\mathbb{C}^{M_B})_\chi \rightarrow (\mathbb{C}^{M_W})_\chi$ is invertible iff $\chi \neq \text{Id}$. Let S_χ denote the inverse operator ($\chi \neq \text{Id}$). We now want to relate S_χ to kernels inverting (continuous) Cauchy-Riemann operators.

Denote by L_χ the holomorphic, unitary line bundle over Σ obtained by twisting the trivial line bundle L_{Id} by the unitary character $\chi : \pi_1(\Sigma) \rightarrow \mathbb{U}$. Sections of L_χ may be seen as multiplicatively multi-valued functions on Σ , with the transformation rule prescribed by χ : $s(\omega + z) = \chi(\omega)s(z)$.

We may consider the operator

$$\bar{\partial}_\chi : L_\chi \rightarrow \Omega^{(0,1)}(L_\chi)$$

canonically associated to the complex structure on L_χ . It is well known that this is a zero index operator which is invertible iff $\chi \neq \text{Id}$. Moreover its inverse has an explicit expression in terms of θ -functions, which we now recall (see eg [16]).

Consider the θ function with characteristics:

$$\vartheta \begin{bmatrix} 2\varepsilon \\ 2\varepsilon' \end{bmatrix} (z) = \sum_{n \in \mathbb{Z}} \exp 2i\pi \left(\frac{1}{2} \tau (n + \varepsilon)^2 + (n + \varepsilon)(z + \varepsilon') \right) = \exp(2i\pi(\frac{\tau}{2}\varepsilon^2 + \varepsilon z + \varepsilon\varepsilon')) \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z + \varepsilon' + \varepsilon\tau)$$

which transforms as:

$$\begin{aligned} \vartheta \begin{bmatrix} 2\varepsilon \\ 2\varepsilon' \end{bmatrix} (z + 1) &= \exp(2i\pi\varepsilon) \vartheta \begin{bmatrix} 2\varepsilon \\ 2\varepsilon' \end{bmatrix} (z) \\ \vartheta \begin{bmatrix} 2\varepsilon \\ 2\varepsilon' \end{bmatrix} (z + \tau) &= \exp(-2i\pi(z + \varepsilon' + \frac{\tau}{2})) \vartheta \begin{bmatrix} 2\varepsilon \\ 2\varepsilon' \end{bmatrix} (z) \end{aligned}$$

(see eg [15]). For concision, let us denote $\vartheta = \vartheta \begin{bmatrix} 2\varepsilon \\ 2\varepsilon' \end{bmatrix}$, $\theta_3 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\theta = \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (these last notations are as in [6]). Then consider the meromorphic function:

$$T(z) = \frac{\vartheta(z)\theta'(0)}{\vartheta(0)\theta(z)}$$

Then $T(z + 1) = -e^{2i\pi\varepsilon}T(z)$, $T(z + \tau) = -e^{-2i\pi\varepsilon'}T(z)$. Moreover $T(z)$ has a simple pole at $z = 0$ with residue 1 (if $(2\varepsilon, 2\varepsilon') \neq (1, 1)$). Thus:

$$S_\chi(z, w) = \frac{1}{\pi} \cdot \frac{\vartheta(z - w)\theta'(0)}{\vartheta(0)\theta(z - w)}$$

is the kernel inverting $\bar{\partial}_\chi$, where χ is the character given by $\chi(1) = -e^{2i\pi\varepsilon}$, $\chi(\tau) = -e^{-2i\pi\varepsilon'}$, $\chi \neq \text{Id}$. We may take $\Im\lambda = \pi(\varepsilon + \frac{1}{2})$, $\Im(\lambda\bar{\tau}) = -\pi(\varepsilon' + \frac{1}{2})$, so that

$$\lambda = \frac{\pi}{\Im\tau} \left(\varepsilon' + \varepsilon\tau + \frac{\tau + 1}{2} \right)$$

Since θ is odd, we have the asymptotic expansion near the diagonal:

$$S_\chi(z, w) = \frac{1}{\pi(z - w)} + r_\chi(w) + O(|z - w|/\vartheta(0))$$

as $z \rightarrow w$, where

$$\pi r_\chi(w) = \lim_{z \rightarrow w} (\pi S_\chi(z - w) - \frac{1}{z - w}) = \frac{\vartheta'}{\vartheta}(0) = 2i\pi\varepsilon + \frac{\theta'_3}{\theta_3}(\varepsilon' + \varepsilon\tau)$$

(which actually depends only on χ). If we set $c_\chi = 2 - \chi(1) - \chi(\tau)$, which is comparable to $(\varepsilon - 1/2)^2 + (\varepsilon' - 1/2)^2$ for $\varepsilon, \varepsilon' \in [0, 1]$, we have $\vartheta(0) = O(\sqrt{c_\chi})$.

If $\varepsilon, \varepsilon'$ depend smoothly on a parameter u , we have:

$$\begin{aligned} \frac{d}{du} \left(\log \vartheta \left[\begin{array}{c} 2\varepsilon \\ 2\varepsilon' \end{array} \right] (0) \right) &= \frac{d}{du} (i\pi\tau\varepsilon^2 + 2i\pi\varepsilon\varepsilon') + \frac{d}{du} \log \theta_3(\varepsilon' + \varepsilon\tau) \\ &= 2i\pi\varepsilon(\dot{\varepsilon}\tau + \dot{\varepsilon}') + 2i\pi\dot{\varepsilon}\varepsilon' + (\dot{\varepsilon}' + \dot{\varepsilon}\tau) \frac{\theta_3'}{\theta_3}(\varepsilon' + \varepsilon\tau) \\ &= (\dot{\lambda}\Im\tau)r_\chi(0) + 2i\pi\dot{\varepsilon}\varepsilon' \end{aligned}$$

the last term being pure imaginary.

Following Ray and Singer ([39]), we consider the *analytic torsion* defined by

$$T_\Sigma(\chi) = \exp\left(-\frac{1}{2}\zeta'_\chi(0)\right) = \det_\zeta((\bar{\partial}_\chi)^*\bar{\partial}_\chi)^{1/2}$$

where ζ_χ is the ζ function defined from the Laplacian-type operator $(\bar{\partial}_\chi)^*\bar{\partial}_\chi$. Then an explicit diagonalisation of $(\bar{\partial}_\chi)^*\bar{\partial}_\chi$ (in flat metric) and Kronecker's second limit formula implies that ([39])

$$T_\Sigma(\chi) = e^{-\pi\nu^2\Im\tau} |\theta(\mu - \tau\nu)/\eta(\tau)|$$

where $\chi(m\tau + n) = \exp(2i\pi(m\mu + n\nu))$ and η is the Dedekind η function (eg [6]). Here $\mu = \frac{1}{2} - \varepsilon'$, $\nu = \varepsilon - \frac{1}{2}$, so that

$$T_\Sigma(\chi) = e^{-\pi(\Im z)^2/\Im\tau} |\theta_3(z)/\eta(\tau)| = |\vartheta \left[\begin{array}{c} 2\varepsilon \\ 2\varepsilon' \end{array} \right] (0)/\eta(\tau)|$$

with $z = \varepsilon' + \varepsilon\tau$. Note that $\left| \vartheta \left[\begin{array}{c} 2\varepsilon \\ 2\varepsilon' \end{array} \right] (0) \right| = e^{-\pi(\Im z)^2/\Im\tau} |\theta_3(z)|$, where $z = \varepsilon' + \varepsilon\tau$.

We admit for now the following near diagonal estimate for S_χ :

$$S_\chi(b, w) - \underline{K}^{-1}(b, w) = i\Im(e^{i\nu}r_\chi(0)) + O(\delta^\varepsilon)$$

for $|b - w| = O(\delta)$ (in particular if $b \sim w$ in M), for some positive constant ε . We also admit that the estimate is uniform in χ for χ in a compact set not containing the identity.

Let us consider now λ as a differentiable function of a parameter u and analyse the variation of $\det K_\lambda$, $K_\lambda : \mathbb{C}^{\Xi_B} \rightarrow \mathbb{C}^{\Xi_w}$. Assume that $\chi_\lambda \neq \text{Id}$, so that K_λ is invertible. Then:

$$\frac{d}{du} \det(K_\lambda) = \text{Tr}(\dot{K}_\lambda K_\lambda^{-1})$$

where for $b \sim w$,

$$\begin{aligned} \dot{K}_\lambda(w, b) &= 2i\Im(\dot{\lambda}(\overline{b-w}))e^{2i\Im(\lambda(\overline{b-w}))}\mathbf{K}(b, w) \\ e^{-2i\Im(\lambda(\overline{b-w}))}K_\lambda^{-1}(b, w) &= S_\chi(b, w) \\ &= \underline{K}^{-1}(b, w) + i\Im(e^{i\nu}r_\chi(0)) + O(\delta/c_\chi) \end{aligned}$$

where $\chi = \chi_\lambda$. If $b \sim b'$ in Γ (resp. Γ^\dagger), w the white vertex corresponding to (bb') , we get $\mathbf{K}(w, b)\underline{K}^{-1}(b, w) = \mathbf{K}(w, b')\underline{K}^{-1}(b', w)$ from Theorem 1. Thus the contribution of $\underline{K}^{-1}(b, w)$ in the trace cancels exactly; besides,

$$2i\Im(\dot{\lambda}(\overline{b-w}))\mathbf{K}(b, w)(i\Im(e^{i\nu}r_\chi(0))) = -\mu_\diamond(w)\Im(\dot{\lambda}e^{-i\nu})\Im(e^{i\nu}r_\chi(0))$$

(where $\nu = \nu(w) + \nu(b)$) and we are left with:

$$\begin{aligned} \text{Tr}(\dot{K}_\lambda K_\lambda^{-1}) &= 2 \sum_{w \in \Xi_w} \mu_\diamond(w) \Re(\dot{\lambda}r_\chi(0)) + O(\delta/c_\chi) = 2 \cdot \text{Area}(\mathbb{C}/\Upsilon) \Re(\dot{\lambda}r_\chi(0)) + O(\delta^\varepsilon) \\ &= 2\Re \frac{d}{du} \log \left| \vartheta \left[\begin{array}{c} 2\varepsilon \\ 2\varepsilon' \end{array} \right] (0) \right| + O(\delta^\varepsilon) = 2 \frac{d}{du} \log T_\Sigma(\chi) + O(\delta^\varepsilon) \end{aligned}$$

since around w , $e^{i\nu(b)}$ is alternatively 1 and i ; and $\text{Area}(\mathbb{C}/\Upsilon) = \Im\tau$. We conclude (admitting the near diagonal estimate) that if $\lambda_i \leftrightarrow \chi_i$ non trivial ($i = 1, 2$), then

$$\frac{\det(K_{\lambda_2})}{\det(K_{\lambda_1})} \longrightarrow \left(\frac{T_{\Sigma}(\chi_2)}{T_{\Sigma}(\chi_1)} \right)^2 = \frac{\det_{\zeta}(\bar{\partial}_{\chi_2})^* \bar{\partial}_{\chi_2}}{\det_{\zeta}(\bar{\partial}_{\chi_1})^* \bar{\partial}_{\chi_1}}$$

as the mesh $\delta \searrow 0$.

5.2 Bosonisation identity

Let ϕ_{nm} be the harmonic differential on Σ with half-integer periods $n = \int_A \phi_{nm}$ and $m = \int_B \phi_{nm}$. Let

$$\begin{aligned} S(\phi) &= 2\pi \int_{\Sigma} |\nabla \phi|^2 dA + \frac{4i\pi}{\Im\tau} \Im((m - \bar{\tau}n)z) + 4i\pi nm \\ &= \frac{2\pi}{\Im\tau} |m - \bar{\tau}n|^2 + 4i\pi(m\varepsilon + n\varepsilon') + 4i\pi nm \end{aligned}$$

where $z = \varepsilon' + \varepsilon\tau = x + iy$, and

$$Z_{inst} = Z_{inst}(z) \stackrel{def}{=} \sum_{n,m \in \frac{1}{2}\mathbb{Z}} e^{-S(\phi_{nm})}$$

Then a Poisson summation argument ([2], Section 4.C) shows that

$$Z_{inst} = (2\Im\tau)^{\frac{1}{2}} e^{-2\pi(\Im z)^2/\Im\tau} |\theta_3(z)|^2$$

The determinants $\det(K_{\lambda})$ count dimer configurations with some unitary weight, which depends only on the periods of the current. This is originally due to Kasteleyn ([23]); see [8] for a recent (and exhaustive) treatment.

We begin with $\det K$ (which is 0 as the kernel of K contains constant functions). Then (for a proper choice of ordering of vertices), each term in the determinant expansion corresponds to a matching of Ξ (with associated current J) counted with a positive sign if $(\int_A J, \int_B J) = (0, 0) \pmod{2}$, and negative sign otherwise. Let us denote this sign by $Q(\mathfrak{m}) = Q(J)$.

In the expansion of $\det K_{\lambda}$, each matching is counted with an additional phase:

$$2\Im \sum_{(bw) \in \mathfrak{m}} \int_w^b \alpha$$

where $\alpha = \lambda d\bar{z}$. As before, we can integrate by parts over a fundamental domain C bounded by cycles A, B drawn on Γ to obtain:

$$\begin{aligned} \sum_{(bw) \in \mathfrak{m}} \int_w^b \alpha &= - \sum_{(bw) \in E_{\Xi}, (ff') = (bw)^{\dagger}} (h(f') - h(f) - \omega_0((ff'))) \int_w^b \alpha \\ &= \int_B dh \int_A \alpha - \int_A dh \int_B \alpha \end{aligned}$$

(Note that in this case, $\sum \omega_0((ff')) \int_b^w \alpha = 0$). Thus:

$$\mathcal{Z}^{-1} \det(K_{\lambda}) = \mathbb{E} \left(Q(J) \exp(2i\Im(-\lambda\bar{\tau} \int_A J + \lambda \int_B J)) \right)$$

where \mathcal{Z} is the partition function of the dimer model on Ξ . Let us set $(n, m) = \frac{1}{2}(\int_A J, \int_B J)$, the half-periods of the current, and

$$\lambda = \lambda(\varepsilon, \varepsilon') = \frac{\pi}{\Im \tau} \left(\varepsilon' + \varepsilon \tau + \frac{\tau + 1}{2} \right)$$

as before. Then:

$$2i\Im(-\lambda\bar{\tau} \int_A J + \lambda \int_B J) = 4i\pi(m\varepsilon + n\varepsilon') + 2i\pi(m + n)$$

Let us remark that

$$Q(J) = -e^{4i\pi(m+\frac{1}{2})(n+\frac{1}{2})} = e^{4i\pi mn} e^{2i\pi(m+n)}$$

so that

$$\mathcal{Z}^{-1} \det(K_\lambda) = \mathbb{E} \left(e^{4i\pi(m\varepsilon + n\varepsilon') + 4i\pi mn} \right)$$

which is 1-periodic in $\varepsilon, \varepsilon'$.

Let us admit for the time being that

$$\frac{\det K_{\lambda_2}}{\det K_{\lambda_1}} \longrightarrow \left(\frac{T_\Sigma(\chi_2)}{T_\Sigma(\chi_1)} \right)^2 = \frac{Z_{inst}(z_2)}{Z_{inst}(z_1)}$$

for $\chi_1, \chi_2 \neq \text{Id}$ (as we have seen, this follows from near diagonal estimates).

This is enough to conclude that (m, n) converges in distribution. Indeed, if we write $K_{\varepsilon, \varepsilon'} = K_{\lambda(\varepsilon, \varepsilon')}$, we have:

$$\mathcal{Z}^{-1} \left(\det K_{\varepsilon, \varepsilon'} + \det K_{\varepsilon+\frac{1}{2}, \varepsilon'} + \det K_{\varepsilon, \varepsilon'+\frac{1}{2}} - \det K_{\varepsilon+\frac{1}{2}, \varepsilon'+\frac{1}{2}} \right) = 2\mathbb{E} \left(e^{4i\pi(m\varepsilon + n\varepsilon')} \right)$$

by virtue of

$$1 + e^{2i\pi m} + e^{2i\pi n} - e^{2i\pi(m+n)} = 2 - (1 - e^{2i\pi m})(1 - e^{2i\pi n}) = 2e^{4i\pi mn}$$

for $m, n \in \frac{1}{2}\mathbb{Z}$. If $(\varepsilon, \varepsilon') = (\frac{1}{2}, \frac{1}{2})$, $\det K_{(\varepsilon, \varepsilon')} = \det K = 0$ (as constant functions are in the kernel of K) and $Z_{inst}(\varepsilon' + \varepsilon\tau) = 0$. Consequently,

$$\mathcal{Z}^{-1}(\det K_{0,0} + \det K_{\frac{1}{2},0} + \det K_{0,\frac{1}{2}}) = 2\mathbb{E}(1) = 2$$

or ([23]):

$$\mathcal{Z} = \frac{1}{2}(\det K_{0,0} + \det K_{\frac{1}{2},0} + \det K_{0,\frac{1}{2}})$$

It follows that, for any $(\varepsilon, \varepsilon') \in \mathbb{R}^2$ (distinguishing the case $(\varepsilon, \varepsilon') \in (\frac{1}{2}\mathbb{Z})^2$, where we use $\det(K) = 0$), we have

$$\mathbb{E} \left(e^{4i\pi(m\varepsilon + n\varepsilon')} \right) \longrightarrow \mathbb{E}_0 \left(e^{4i\pi(m\varepsilon + n\varepsilon')} \right)$$

where \mathbb{E}_0 is relative to the probability measure on $(\frac{1}{2}\mathbb{Z})^2$ with weights proportional to $e^{-\frac{2\pi}{\Im \tau}|m - \bar{\tau}n|^2}$. Convergence of the characteristic function then implies convergence in distribution of the half-periods (n, m) .

5.3 General Cauchy-Riemann operators

Let $\alpha = a(z)d\bar{z}$ be an Υ -periodic $(0, 1)$ -form (equivalently, a $(0, 1)$ -form on Σ). We can define a perturbed operator

$$K_\alpha(w, b) = K(w, b) \exp(2i\Im \int_w^b \alpha)$$

so that K_α defines an operator $\mathbb{C}^{M_B} \rightarrow \mathbb{C}^{M_W}$ and by quotienting an operator $\mathbb{C}^{\Xi_B} \rightarrow \mathbb{C}^{\Xi_W}$. We are concerned with the inverse of that last operator. We first consider the limiting continuous Cauchy-Riemann operators.

The Dolbeault decomposition of α reads:

$$\alpha = \lambda_0 d\bar{z} + \bar{\partial}g$$

where g is a function on Σ , $\lambda_0 \in \mathbb{C}$ constant. Thus $\bar{\partial} + \alpha = e^{-g}(\bar{\partial} + \lambda_0 d\bar{z})e^g$. In the case $\chi = \chi_{\lambda_0} \neq \text{Id}$, this is invertible, with inverting kernel given by

$$S_\alpha(b, w) = S_\chi(b, w)e^{g(w)-g(b)}$$

From the expansion $S_\chi(b, w)e^{2i\Im\lambda_0\overline{(b-w)}} = 1/\pi(b-w) + r_\chi + O(|z-w|/\sqrt{c_\chi})$, we obtain the asymptotic expansion as $b \rightarrow w$, $w \in \Sigma$ fixed, $\lambda = a(w)$:

$$\begin{aligned} S_\alpha(b, w)e^{2i\Im\lambda\overline{(b-w)}} &= \left(\frac{1}{\pi(b-w)} + r_\chi + O(|b-w|/\sqrt{c_\chi}) \right) e^{2i\Im g_\Sigma(w)\overline{(b-w)} - (g(b)-g(w))} \\ &= \frac{1}{\pi(b-w)} + r_\chi - \frac{1}{\pi}(\bar{g}_z(w) + g_z(w)) + O(|b-w|\cdot\|g\|_{C^2}/\sqrt{c_\chi}) \end{aligned}$$

Let us turn now to the discrete operator K_α , seen as a finite difference operator. For a function ϕ on Σ , we define as before restriction operators R_B, \bar{R}_B by: $(R_B\phi)(b) = (\bar{R}_B\phi)(b) = \phi(b)$ for $b \in \Gamma$ and $(R_B\phi)(b) = -(\bar{R}_B\phi)(b) = i\phi(b)$ for $b \in \Gamma^\dagger$. Let $w \in M_W$ be an edge node, with black neighbours x, y, x', y' in ccwise order labelled in such a way that (xx') is an oriented edge of Γ . We denote $(R_W\beta)(w) = b(w)e^{-i\nu(w)}$ for $\beta = b(z)d\bar{z}$; then (see (4.4))

$$K_\alpha(R_B\phi) = 2\mu_\diamond R_W(\bar{\partial}\phi + \phi\alpha) + O(\delta^4\|\phi\|_{C^3}(1 + \|a\|_{C^2})^3)$$

Similarly, with $(\bar{R}_W\beta) = b(w)e^{i\nu(w)}$ for $\beta = b(z)dz$, we have:

$$K_\alpha(\bar{R}_B\phi) = 2\mu_\diamond \bar{R}_W(\partial\phi - \phi\bar{\alpha}) + O(\delta^4\|\phi\|_{C^3}(1 + \|a\|_{C^2})^3)$$

Fix $w_0 \in M_W$ and set $\lambda = a(w_0)$. Then:

$$K_\alpha(w, b) = e^{2i\Im(\bar{\lambda}w)}\mathbf{K}(w, b)e^{-2i\Im(\bar{\lambda}b)} + O(\delta^2 \text{dist}(b, w_0)\|a\|_{C^1})$$

Set

$$\tilde{S}_\alpha(b, w) = e^{2i\Im(\bar{\lambda}(b-w))}(\underline{\mathbf{K}}^{-1}(b, w) + R_B(\mu) + \bar{R}_B(\mu'))$$

for $|b-w| < \eta$ and

$$\tilde{S}_\alpha(b, w) = \frac{1}{2} \left(R_B(e^{i\nu(w)}S_\alpha(b, w)) + \bar{R}_B(e^{-i\nu(w)}\overline{S_{-\alpha}(b, w)}) \right)$$

for $|b-w| \geq \eta$, where η is a mesoscopic scale and

$$\begin{aligned} \mu &= \frac{1}{2}e^{i\nu(w)}(r_\chi - \frac{2}{\pi}(\partial_z \Re g)(w)) \\ \mu' &= \frac{1}{2}e^{-i\nu(w)}(\bar{r}_\chi + \frac{2}{\pi}(\partial_{\bar{z}} \Re g)(w)) = -\bar{\mu} \end{aligned}$$

Reasoning as in the planar case (it is a bit simpler here as Σ is compact), we may use this approximate inverse to show that K_α is invertible for δ small enough and its inverse S_α satisfies the following near diagonal estimate:

$$S_\alpha(b, w) - e^{2i\Im\bar{\lambda}(b-w)}\underline{\mathbf{K}}^{-1}(b, w) = i\Im(e^{i\nu}r_\alpha) + O(\delta^\varepsilon)$$

where $r_\alpha = r_\chi - \frac{2}{\pi}(\partial_z \Re g)(w)$, ε is a positive constant, and the error term is uniform in (λ_0, g) for $\|g\|_{C^2}$ bounded and $\chi = \chi_{\lambda_0}$ in a compact subset of the group of characters not containing the identity (r_χ blows up as $\chi \rightarrow \text{Id}$).

Similarly to the planar case, this implies:

$$\frac{\det K_{\alpha_2}}{\det K_{\alpha_1}} \exp(2i(P(i\alpha_2) - P(i\alpha_1))) \longrightarrow \exp\left(-\frac{1}{2\pi} \left(\int_{\Sigma} |\nabla \Re g_2|^2 dA - \int_{\Sigma} |\nabla \Re g_1|^2 dA \right)\right) \left(\frac{T_{\Sigma}(\chi_2)}{T_{\Sigma}(\chi_1)} \right)^2$$

where $\alpha_i = \lambda_i d\bar{z} + \bar{\partial}g_i$, χ_i is the associated character, and convergence is uniform for $\|g_i\|_{C^2}$ bounded and χ_i away from the trivial character. Let us point out at this stage that the additive decomposition of r_{α} reflecting the Dolbeault decomposition of α leads to the multiplicative decomposition of the relative determinant $\det(K_{\alpha_2})/\det(K_{\alpha_1})$ (in the fine mesh limit), which in turns yields the independence of the instanton and scalar components of the limiting field.

To complete the identification, we notice that

$$\det(K_{\alpha}) \exp(2iP(i\alpha)) = \mathcal{Z} \mathbb{E} \left(\exp(4i\pi(m\varepsilon + n\varepsilon') + 4i\pi mn) \exp(-i\Re \int_{\Sigma} h(\Delta g) dA) \right)$$

and

$$\mathcal{Z} = \frac{1}{2} (\det K_{0,0} + \det K_{\frac{1}{2},0} + \det K_{0,\frac{1}{2}})$$

which implies

$$\mathbb{E} \left(\exp(4i\pi(m\varepsilon + n\varepsilon')) \exp(-i\Re \int_{\Sigma} h(\Delta g) dA) \right) \longrightarrow \mathbb{E}_0 \left(\exp(4i\pi(m\varepsilon + n\varepsilon')) \right) \exp \left(-\frac{1}{2\pi} \int_{\Sigma} |\nabla \Re g|^2 dA \right)$$

in the generic case $(\varepsilon, \varepsilon') \notin (\frac{1}{2}\mathbb{Z})^2$. In the case $g = 0$, we already noticed that this ensures convergence in distribution of the half-periods (n, m) . Then for $(n_0, m_0) \in (\frac{1}{2}\mathbb{Z})^2$, we write

$$\delta_{(n_0, m_0)}(n, m) = \int_{[0,1]^2} e^{4i\pi((m-m_0)\varepsilon + (n-n_0)\varepsilon')} d\varepsilon d\varepsilon'$$

and

$$\mathbb{E} \left(\mathbf{1}_{\{(n,m)=(n_0,m_0)\}} \exp(-i\Re \int_{\Sigma} h(\Delta g) dA) \right) = \int_{[0,1]^2} e^{-4i\pi(m_0\varepsilon + n_0\varepsilon')} \mathbb{E} \left(e^{4i\pi(m\varepsilon + n\varepsilon')} \exp(-i\Re \int_{\Sigma} h(\Delta g) dA) \right) d\varepsilon d\varepsilon'$$

Since $\mathbb{P}((n, m) = (n_0, m_0)) \rightarrow \mathbb{P}_0((n, m) = (n_0, m_0)) > 0$, we deduce by dominated convergence:

$$\mathbb{E} \left(\exp(-i\Re \int_{\Sigma} h(\Delta g) dA) \middle| (n, m) = (n_0, m_0) \right) \longrightarrow \exp \left(-\frac{1}{2\pi} \int_{\Sigma} |\nabla \Re g|^2 dA \right)$$

for any $(n_0, m_0) \in (\frac{1}{2}\mathbb{Z})^2$.

From this we deduce:

Proposition 5 (Finite dimensional marginals). *Let $\alpha_1, \dots, \alpha_k$ be C^1 1-forms on Σ . As $\delta \searrow 0$, the joint distribution of*

$$\left(\int_{\Sigma} J_{\delta} \wedge * \alpha_1, \dots, \int_{\Sigma} J_{\delta} \wedge * \alpha_k \right)$$

converges to the joint distribution of

$$\left(\int_{\Sigma} J \wedge * \alpha_1, \dots, \int_{\Sigma} J \wedge * \alpha_k \right)$$

*where J is the current of the compactified free field on Σ with compactification radius 1 and action $S(J) = \frac{\pi}{2} \int_{\Sigma} J \wedge * J$.*

Proof. We have the Hodge decomposition $\alpha_j = (a_j dx + b_j dy) + dg_j$, $g_j \in C^2$. We know that the marginal distribution of the current periods converges, and that the conditional characteristic function

$$\mathbb{E}(\exp(i \sum_j \lambda_j \int_{\Sigma} J \wedge * dg_j) | (n, m))$$

converges pointwise. This is enough to ensure convergence of the joint distribution. \square

To obtain a functional CLT from this finite dimensional CLT, we need a tightness estimate.

Lemma 6. *For any $\varepsilon > 0$, the family of probability measures induced on $H^{-2-\varepsilon}(\Omega^1(\Sigma))$ by the current J_{δ} is tight.*

Proof. Let us remark that for a r.v. X ,

$$\mathbb{E}(X^2) \leq \mathbb{E}(Q_{0,0}X^2) + \mathbb{E}(Q_{0,1}X^2) + \mathbb{E}(Q_{1,0}X^2)$$

where $Q_{\varepsilon,\varepsilon'} = e^{4i\pi(m\varepsilon+n\varepsilon')+4i\pi mn}$ (a random sign), since $1 \leq Q_{0,0} + Q_{0,1} + Q_{1,0} \leq 3$.

Set $\alpha_{(\varepsilon,\varepsilon')}(t) = \lambda(\varepsilon,\varepsilon')d\bar{z} + t\bar{\partial}g$ and $X = \Re \int h\Delta g dA$. We get

$$\mathbb{E}((\Re \int h_0 \Delta g dA)^2) \leq -Z^{-1} \frac{d^2}{dt^2} \Big|_{t=0} (\det K_{\alpha_{0,0}(t)} + \det K_{\alpha_{0,1}(t)} + \det K_{\alpha_{1,0}(t)})$$

We have:

$$\det K_{\alpha_{\varepsilon,\varepsilon'}(t)} = Z \mathbb{E}(Q_{\varepsilon,\varepsilon'} e^{-itX}) \exp(-2itP(i\bar{\partial}g))$$

and thus

$$\mathbb{E}(Q_{\varepsilon,\varepsilon'} X^2) = -2Z^{-1} \frac{d^2}{dt^2} \Big|_{t=0} (\det K_{\alpha_{\varepsilon,\varepsilon'}(t)} \exp(2itP(i\bar{\partial}g)))$$

We now assume that $(\varepsilon, \varepsilon') \in \{(0,0), (0,1), (1,0)\}$, so that in particular $\det K_{\alpha_{\varepsilon,\varepsilon'}(t)}$ is invertible at $t = 0$, and

$$\begin{aligned} \frac{d}{dt} \log \mathbb{E}(Q_{\varepsilon,\varepsilon'} e^{itX}) &= \text{Tr}(K_{\alpha}^{-1} \dot{K}_{\alpha}) + 2iP(i\bar{\partial}g) \\ \frac{d^2}{dt^2} \log \mathbb{E}(Q_{\varepsilon,\varepsilon'} e^{itX}) &= \text{Tr}(K_{\alpha}^{-1} \ddot{K}_{\alpha}) - \text{Tr}((K_{\alpha}^{-1} \dot{K}_{\alpha})^2) \end{aligned}$$

where $\alpha = \alpha_{\varepsilon,\varepsilon'}(t)$ for brevity.

From the short distance asymptotics for K_{α}^{-1} , we get

$$\text{Tr}(K_{\alpha}^{-1} \dot{K}_{\alpha}) + 2iP(i\bar{\partial}g) = O(\delta^{\varepsilon''} \|g\|_{C^1})$$

at $t = 0$ for some $\varepsilon'' > 0$. We also have

$$\text{Tr}(K_{\alpha}^{-1} \ddot{K}_{\alpha}) = O(\|g\|_{C^1})$$

We are left with estimating $\text{Tr}((K_{\alpha}^{-1} \dot{K}_{\alpha})^2)$ (at $t = 0$). Since $K_{\alpha}^{-1}(b, w) = \underline{K}^{-1}(b, w) + O(1)$ and $\dot{K}_{\alpha} = O(\delta^2(1 + \|g\|_{C^1}))$, by isolating the leading singularity we may write

$$\text{Tr}((K_{\alpha}^{-1} \dot{K}_{\alpha})^2) = \sum_{|w-w'| \leq \varepsilon_0} (\dot{K}_{\alpha} \underline{K}^{-1})(w, w') (\dot{K}_{\alpha} \underline{K}^{-1})(w', w) + O((1 + \|g\|_{C^1})^2)$$

where $\varepsilon_0 \leq \min(1, \Im\tau)/10$, say. Fix w ; replacing K_α with K_β where $\beta = tg_{\bar{z}}(w)d\bar{z}$ (a constant $(0,1)$ -form) induces an error of order $O(\|g\|_{C^{1,\varepsilon}}(1 + \|g\|_{C^1}))$ (here $C^{1,\varepsilon}$ designates functions with ε -Hölder derivative, $\varepsilon > 0$ arbitrarily small but fixed). Thus we simply need to estimate

$$\sum_{w': |w-w'| \leq \varepsilon_0} (\dot{K}_\beta \underline{K}^{-1})(w, w') (\dot{K}_\beta \underline{K}^{-1})(w', w)$$

which may be thought of as a discrete version of a principal value integral of type $p.v. \iint \frac{dA(w')}{(w'-w)^2}$. A probabilistic interpretation of this quantity goes as follows: a product $\underline{K}^{-1}(b, w') \underline{K}^{-1}(b', w)$ (for $b \sim w$, $b' \sim w'$) is, up to multiplicative local factors, the covariance $\text{Cov}(\mathbf{1}_{(bw) \in \mathfrak{m}}, \mathbf{1}_{(b'w') \in \mathfrak{m}})$ under the appropriate Gibbs measure on tilings of the full plane. By linearity (w is fixed and we sum over w'), we are left with estimating $\text{Cov}(\mathbf{1}_{(bw) \in \mathfrak{m}}, \ell)$, where ℓ is a linear function of the heights in $B(w, \varepsilon_0)$. Since β is constant, $\partial\beta = 0$ and it follows that ℓ depends only on the heights on $\partial B(w, \varepsilon_0)$. Since $\text{Cov}(\mathbf{1}_{(bw) \in \mathfrak{m}}, \mathbf{1}_{(b'w') \in \mathfrak{m}}) = O(1/|w' - w|^2)$, we conclude:

$$\sum_{w': |w-w'| \leq \varepsilon_0} (\dot{K}_\beta \underline{K}^{-1})(w, w') (\dot{K}_\beta \underline{K}^{-1})(w', w) = O(\delta^2 \|g\|_{C^1}^2)$$

We finally get the estimate

$$\mathbb{E}((\Re \int h_0 \Delta g dA)^2) = O((1 + \|g\|_{C^1})(1 + \|g\|_{C^{1,\varepsilon}}))$$

which is uniform in δ for δ small enough (the limit as $\delta \searrow 0$ is of order $\|g\|_{C^1}^2$).

Consider an eigenbasis for the Laplacian on $\Sigma = \mathbb{C}/\Upsilon$: set

$$g_u(z) = \exp(i\Re(z\bar{u}))$$

where $u \in \check{\Upsilon} = \{v \in \mathbb{C} : \forall z \in \Upsilon, \Re(z\bar{v}) \in 2\pi\mathbb{Z}\}$. Note that $\|g_u\|_{C^k} = O(1 + |u|^k)$ and $\|g_u\|_{C^{1,\varepsilon}} = O(1 + |u|^{1+\varepsilon})$. We may define

$$\left\| \sum_{u \in \check{\Upsilon}} a_u g_u \right\|_{H^s}^2 = \sum_{u \in \check{\Upsilon}} |a_u|^2 (1 + |u|^2)^s$$

If we choose h_0 (which is given modulo an additive constant) so that $\int_\Sigma h_0 dA = 0$, we may write $h_0 = \sum_{u \in \check{\Upsilon} \setminus \{0\}} a_u g_u$, where

$$\mathbb{E}((a_u)^2) \leq c|u|^{\varepsilon-2}$$

for $u \in \check{\Upsilon} \setminus \{0\}$, and c is uniform in δ, u . Consequently, the Chebychev inequality yields:

$$\mathbb{P}(\forall u \in \check{\Upsilon} \setminus \{0\}, |a_u| \leq C|u|^{-\gamma}) \geq 1 - \frac{c}{C^2} \sum_{u \in \check{\Upsilon} \setminus \{0\}} |u|^{\varepsilon-2-2\gamma}$$

where the sum converges if $2\gamma + \varepsilon < 0$. On this event,

$$\left\| \sum_u a_u g_u \right\|_{H^s}^2 \leq c \sum_{u \in \check{\Upsilon} \setminus \{0\}} |u|^{2s-2\gamma}$$

which converges if $2s - 2\gamma < -2$. By taking $\gamma \in (-\varepsilon, -\varepsilon/2)$, and observing that $H^{s_1}(\Sigma)$ is compactly embedded in $H^{s_2}(\Sigma)$ for $s_1 > s_2$, we conclude that the probability measures induced by h_0 on $H^{-1-\varepsilon}(\Sigma)$ are tight for δ small enough.

The current can be decomposed as $J_\delta = dh_0 + \omega_h$. Since ω_h takes values in a two-dimensional lattice of $H^s(\Omega^1(\Sigma))$ and converges in distribution, it induces a tight family of probability measures (s arbitrary). The lemma follows. □

We may now state a functional limit theorem for the current.

Theorem 7. *For any $\varepsilon > 0$, the probability measures induced on $H^{-2-\varepsilon}(\Omega^1(\Sigma))$ by the current J_δ converge as $\delta \searrow 0$ to the distribution of the current of the compactified free field on Σ with compactification radius 1 and action functional $S(J) = \frac{\pi}{2} \int J \wedge *J$.*

Proof. We have obtained tightness of the measures and convergence of the characteristic functional. The dual of $H^{-2-\varepsilon}(\Omega^1(\Sigma))$ may be identified with $H^{2+\varepsilon}(\Omega^1(\Sigma))$, which consists of 1-forms with C^1 coefficients. Indeed, $H^{2+\varepsilon}$ may be embedded in $W^{2,p+\varepsilon'}$ (eg [1], 7.58), the elements of which have continuous versions with ε'' -Hölder derivatives (Morrey). For Banach space-valued variables, tightness and pointwise convergence of the characteristic functional ensures weak convergence (eg [34], 0.2.1). \square

6 Surgery

In the most general set-up, we shall be interested with correlations of electric and magnetic operators for the height function in the plane, on a torus or in a finite planar domains. This corresponds to a certain Cauchy-Riemann problem, with singularities at insertions. In this section, we show how one can treat separately issues stemming from boundary conditions, electric insertions and magnetic insertions and then glue these components together.

We start from a family of rhombi tiling $(\Lambda_\delta)_\delta$ with edge mesh δ going to zero along some sequence. Let D_i (resp. D_o) be a simply connected neighbourhood of 0 (resp. ∞) in $\hat{\mathbb{C}}$ that intersect in an annular domain A . Let γ be a simple closed loop such that A is a tubular neighbourhood of γ . We assume that the distance between γ and ∂A is large enough compared with δ .

Let $\Xi_i = \Xi_i(\delta)$ (resp. Ξ_o) be a bipartite graph that agrees with M in D_o (resp. D_i); $K_s : \mathbb{C}^{\Xi_s^B} \rightarrow \mathbb{C}^{\Xi_s^W}$, $s \in \{i, o\}$, is a linear operator such that K_i (resp. K_o) agrees with K in D_o (resp. D_i); in other words, Ξ_i, K_i are obtained by modifying M, K in $D_i \setminus D_o$, and vice versa). We assume that K_s is invertible in that for each $w \in \Xi_s^W$, there is a unique function $f \in \mathbb{C}^{\Xi_s^B}$ vanishing at infinity such that $K_s f = \delta_w$, $s \in \{i, o\}$; it is denoted $K_s^{-1}(\cdot, w)$. This implies in particular that $K_s f = 0$ and f vanishes at infinity iff $f = 0$. We assume that, for $s \in \{i, o\}$, there are kernels $(z, w) \mapsto S_s(z, w)$, $(z, w) \mapsto \bar{S}_s(z, w)$, and η is an error rate: $\lim_{\delta \searrow 0} \eta(\delta) = 0$ such that:

$$K_s^{-1}(b, w) = \frac{1}{2} R_B(e^{i\nu(w)} S_s(b, w)) + \frac{1}{2} \overline{R_B}(e^{-i\nu(w)} \bar{S}_s(b, w)) + O(\eta(\delta))$$

on $A^2 \setminus \Delta_A$, uniformly on compact subsets, where $\Delta_A = \{(x, y) \in A^2 : x \neq y\}$. Note that we do not request that $\bar{S}_s(z, w) = \overline{S_s(z, w)}$, which is the case when K_s is real. More precisely, we assume that

$$(z, w) \mapsto T_s(z, w) = S_s(z, w) - \frac{1}{\pi(z-w)}$$

is C^1 in A^2 and holomorphic in z in A (in the presence of boundaries, it will be harmonic, rather than holomorphic, in w). The holomorphicity condition is actually superfluous but will always be obvious in applications. Correspondingly $(z, w) \mapsto \bar{T}_s(z, w) = \bar{S}_s(z, w) - \frac{1}{\pi(\bar{z}-\bar{w})}$ is C^1 in (z, w) and antiholomorphic in z .

A discrete Cauchy integral formula argument (see below) shows that the convergence assumption is equivalent to assuming:

$$K_s^{-1}(b, w) = \underline{K}^{-1}(b, w) + \frac{1}{2} R_B(e^{i\nu(w)} T_s(b, w)) + \frac{1}{2} \overline{R_B}(e^{-i\nu(w)} \bar{T}_s(b, w)) + O(\eta(\delta))$$

on A^2 , uniformly on compact subsets (including on the diagonal).

We now consider the glued data: Ξ_g agrees with Ξ_s in D_s , $s \in \{i, o\}$; $K_g : \mathbb{C}^{\Xi_g^B} \rightarrow \mathbb{C}^{\Xi_g^W}$ agrees with K_s in D_s , $s \in \{i, o\}$. We wish to estimate K_g^{-1} (if defined).

Let us consider γ_δ a simple cycle on Γ (and thus on M) which approximates γ in the sense that each arc of γ of length ℓ is at Hausdorff distance $\leq C\delta$ of an arc of γ_δ of length at most $C\ell$, $C > 0$ fixed.

Let $f \in \mathbb{C}^{\Xi_i^B}$ be such that $K_i f(w) = 0$ for any $w \in \Xi_i^W$ which is (strictly) inside γ_δ . Let $\hat{f}(b) = f(b)$ if b is on or inside γ_δ and 0 otherwise. Then $K_i \hat{f}$ is supported on white vertices on γ_δ or adjacent to a vertex on γ_δ . Moreover $\hat{f} - K_i^{-1}(K_i \hat{f}) = 0$ (since it is in the kernel of K_i and vanishes at infinity). This yields the Cauchy integral formula:

$$f(b) = \sum_w K_i^{-1}(b, w)(K_i \hat{f})(w)$$

for b on or within γ_δ . Let γ^B be the set of black vertices which are either on γ_δ (and thus on Γ) or inside of γ_δ and adjacent to a white vertex on γ_δ (and thus on Γ^\dagger). Correspondingly, let γ^W be the set of white vertices which are either on γ_δ or outside of γ_δ and adjacent to a black vertex on γ_δ . Let $K_\gamma(w, b) = K(w, b)$ if $w \in \gamma^W$, $b \in \gamma^B$ and $K_\gamma(w, b) = 0$ otherwise. We can rephrase the Cauchy formula as:

$$f(b) = \sum_{w \in \gamma^W} K_i^{-1}(b, w)(K_\gamma f|_{\gamma^B})(w)$$

for $b \in \Xi_i^B$ on or inside γ_δ and f such that $(K_i f)(w) = 0$ for w inside γ_δ . The (discrete) *Cauchy data space* (on γ , for K_i) is the subspace $C_i^\delta \subset \mathbb{C}^{\gamma^B}$ consisting of restrictions to γ^B of functions f such that $K_i f = 0$ strictly inside γ_δ . Clearly,

$$P_i^\delta : \mathbb{C}^{\gamma^B} \longrightarrow \mathbb{C}^{\gamma^B}$$

$$g \longmapsto \sum_{w \in \gamma^W} K_i^{-1}(., w)(K_\gamma g)(w)$$

is a projector onto the Cauchy data space C_i^δ . We want to relate this to limiting continuous Cauchy data spaces. Given our data, we can define them in terms of the following operators on functions on γ :

$$P_i(f)(z_0) = \lim_{z \rightarrow z_0} \frac{1}{2i} \oint_\gamma S_i(z, w) f(w) dw$$

$$\bar{P}_i(f)(z_0) = \lim_{z \rightarrow z_0} -\frac{1}{2i} \oint_\gamma \bar{S}_i(z, w) f(w) d\bar{w}$$

where limits are taken from inside γ . Writing

$$\frac{1}{2i} \oint_\gamma S_i(z, w) f(w) dw = \frac{1}{2i} \oint_\gamma T_i(z, w) f(w) dw + f(z) + \oint_\gamma \frac{f(w) - f(z)}{2i\pi(z - w)} dw$$

gives that P_i is a bounded operator $\text{Lip}(\gamma) \rightarrow C^0(\gamma)$. This defines Cauchy data spaces by $C_i = \{f \in \text{Lip}(\gamma) : f = P_i f\}$, $\bar{C}_i = \{f \in \text{Lip}(\gamma) : f = \bar{P}_i f\}$ (more classically one would consider the L^2 closure of these; Lipschitz functions are sufficient for our purposes).

Let f be Lipschitz in a neighbourhood of γ . We wish to estimate the discrete ‘‘contour integral’’

$$\sum_{w \in \gamma^W} K_i^{-1}(b, w)(K_\gamma(R_B f))(w)$$

where $(R_B f)(b) = f(b)$ if $b \in \Gamma \cap \gamma^B$ and $(R_B f)(b) = if(b)$ if $b \in \Gamma^\dagger \cap \gamma^B$. For this we note that γ^W consists of white vertices on γ_δ (a simple cycle on Γ) and white vertices on γ_δ^\dagger (a cycle on Γ^\dagger). The path γ_δ^\dagger may be described as follows. Let $(b_0, b_1, \dots, b_n = b_0)$ be the black vertices on γ_δ (taken counterclockwise).

For each i , enumerate (in counterclockwise order) the faces of Γ which are adjacent to b_i and outside of γ_δ : $b_{i,1}^\dagger, \dots, b_{i,k_i}^\dagger$. Concatenating these lists, one gets $\gamma_\delta^\dagger = (b_{0,1}^\dagger, \dots, b_{0,k_0}^\dagger, b_{1,1}^\dagger, \dots, b_{n-1,k_{n-1}}^\dagger, \dots, b_{n,1}^\dagger = b_{0,1}^\dagger)$, a cycle on Γ^\dagger (which may involve some backtracking). Without loss of generality, one may assume that the reference orientation (used to define the Kasteleyn orientation) of edges of Γ on γ_δ agrees with the direct orientation of γ_δ . Then if w on γ_δ corresponds to the edge (bb') of Γ ,

$$(\mathbf{K}_\gamma(R_B f))(w) = \frac{1}{2}|b' - b|f(b) + O(\delta^2 \|f\|_{Lip})$$

and if w on γ_δ^\dagger corresponds to the edge (bb') of Γ^\dagger ,

$$(\mathbf{K}_\gamma(R_B f))(w) = \text{sgn}(\mathbf{K}(b_0, w)) \frac{1}{2}|b' - b|f(b) + O(\delta^2 \|f\|_{Lip})$$

and b_0 is the black neighbour of w which is on γ^\dagger (if w has two neighbours on γ^\dagger , $(\mathbf{K}_\gamma(R_B f))(w) = O(\delta^2 \|f\|_{Lip})$). Then

$$\sum_{w \in \gamma_W} (e^{i\nu(w)} T_i(z, w)) (\mathbf{K}_\gamma(R_B f))(w) = \frac{i}{2} \oint_{\gamma_\delta} T_i(z, w) f(w) dw + \frac{i}{2} \oint_{\gamma_\delta^\dagger} T_i(z, w) f(w) dw + O(\delta \|f\|_{Lip} \|S_i\|_\infty)$$

and by Stokes' formula, $\oint_{\gamma_\delta} T_i(z, w) f(w) dw - \oint_{\gamma_\delta^\dagger} T_i(z, w) f(w) dw = O(\delta \|f\|_{Lip} \|T_i\|_\infty)$ (as the area of the annulus between $\gamma_\delta, \gamma_\delta^\dagger$ is $O(1)$). (Estimates are uniform for z in a compact subset of A). Similarly,

$$\begin{aligned} \sum_{w \in \gamma_W} (e^{i\nu(w)} T_i(z, w)) (\mathbf{K}_\gamma(R_B f))(w) &= i \oint_{\gamma} T_i(z, w) f(w) dw + O(\delta \|f\|_{Lip} \|S_i\|_\infty) \\ \sum_{w \in \gamma_W} (e^{i\nu(w)} T_i(z, w)) (\mathbf{K}_\gamma(\overline{R_B f}))(w) &= O(\delta \|f\|_{Lip} \|T_i\|_\infty) \\ \sum_{w \in \gamma_W} (e^{-i\nu(w)} \bar{T}_i(z, w)) (\mathbf{K}_\gamma(R_B f))(w) &= O(\delta \|f\|_{Lip} \|T_i\|_\infty) \\ \sum_{w \in \gamma_W} (e^{-i\nu(w)} \bar{T}_i(z, w)) (\mathbf{K}_\gamma(\overline{R_B f}))(w) &= -i \oint_{\gamma} \bar{T}_i(z, w) f(w) d\bar{w} + O(\delta \|f\|_{Lip} \|T_i\|_\infty) \end{aligned}$$

To deal with the singular part, we observe that the constant functions $R_B(\mu)$, $\overline{R_B}(\mu)$ are discrete holomorphic and consequently by replication:

$$\begin{aligned} \sum_{w \in \gamma_W} \underline{\mathbf{K}}^{-1}(b, w) (\mathbf{K}_\gamma(R_B(\mu)))(w) &= R_B(\mu) \mathbf{1}_{\gamma^i}(b) \\ \sum_{w \in \gamma_W} \underline{\mathbf{K}}^{-1}(b, w) (\mathbf{K}_\gamma(\overline{R_B}(\mu)))(w) &= \overline{R_B}(\mu) \mathbf{1}_{\gamma^i}(b) \end{aligned}$$

where γ^i is the set of vertices on or inside γ_δ . Then

$$\begin{aligned} \sum_{w \in \gamma_W} \underline{\mathbf{K}}^{-1}(b, w) (\mathbf{K}_\gamma(R_B f))(w) &= R_B(f(b)) \mathbf{1}_{\gamma^i} + \sum_{w \in \gamma_W} \underline{\mathbf{K}}^{-1}(b, w) (\mathbf{K}_\gamma(R_B(f - f(b))))(w) \\ &= R_B(f(b)) \mathbf{1}_{\gamma^i} + \frac{i}{2\pi} R_B \left(\oint_{\gamma} \frac{f(w) - f(b)}{b - w} dw \right) + O(\delta |\log \delta| \|f\|_{Lip}) \end{aligned}$$

and similarly

$$\sum_{w \in \gamma_W} \underline{\mathbf{K}}^{-1}(b, w) (\mathbf{K}_\gamma(\overline{R_B f}))(w) = \overline{R_B}(f(b)) \mathbf{1}_{\gamma^i} - \frac{i}{2\pi} \overline{R_B} \left(\oint_{\gamma} \frac{f(w) - f(b)}{\bar{b} - \bar{w}} d\bar{w} \right) + O(\delta |\log \delta| \|f\|_{Lip})$$

We conclude that if f is Lipschitz around γ ($\|f\|_{Lip}$ its Lipschitz norm in a neighbourhood of γ), then

$$\|P_i^\delta(R_B(f)) - R_B(P_i f)\|_\infty = O(\tilde{\eta}(\delta)\|f\|_{Lip})$$

(on γ), where $\tilde{\eta}(\delta) = \eta(\delta) \vee (\delta|\log \delta|)$. In particular, if f is Lipschitz around γ , then it is in the (continuous) Cauchy data space C_i iff $\|P_i^\delta(R_B(f)) - R_B(P_i f)\|_\infty$ goes to zero as $\delta \searrow 0$ (uniform norm on γ^B). Correspondingly, with the same assumptions,

$$\|P_i^\delta(\overline{R_B(f)}) - \overline{R_B(P_i f)}\|_\infty = O(\tilde{\eta}(\delta)\|f\|_{Lip})$$

again on γ .

Having analysed *inner* Cauchy data spaces, one may repeat the argument for *outer* Cauchy data spaces. Let γ^B be the black vertices which are on γ_δ or are outside of γ_δ and adjacent to a white vertex of γ_δ . The outer Cauchy data space $C_o^\delta \subset \mathbb{C}^{\gamma_o^B}$ consists of restrictions to γ_o^B of functions $f \in \mathbb{C}^{\Xi_o^B}$ such that $K_o f(w) = 0$ for any $w \in \Xi_o^W$ strictly outside of γ_δ and f vanishes at infinity. The continuous outer Cauchy data space C_o, \bar{C}_o are defined as fixed points of the operators P_o, \bar{P}_o :

$$\begin{aligned} P_o(f)(z_0) &= \lim_{z \rightarrow z_0} \frac{1}{2i} \oint_\gamma S_o(z, w) f(w) dw \\ \bar{P}_o(f)(z_0) &= \lim_{z \rightarrow z_0} -\frac{1}{2i} \oint_\gamma \bar{S}_o(z, w) f(w) d\bar{w} \end{aligned}$$

where limits are taken from outside γ .

Let us address uniqueness for K_g , ie $K_g f = 0$, f vanishing at infinity implies that $f = 0$. This will follow (for small enough δ) from the following natural assumption on continuous Cauchy data spaces: $C_i \cap C_0 = \{0\}$, $\bar{C}_i \cap \bar{C}_0 = \{0\}$. Indeed, assume by contradiction that for some sequence $\delta_n \searrow 0$, there is $f_n \in \mathbb{C}^{\Xi_g^{B, \delta_n}} \neq 0$ such that $K_g^{\delta_n} f_n = 0$, f_n vanishes at infinity (the line of argument here is similar to some arguments in [7]). Let γ be a simple cycle in A that disconnects 0 from ∞ . We normalise f_n so that $\|(f_n)|_\gamma\|_\infty = 1$. Let γ_1, γ_2 be two disjoint cycles in A bounding an open annulus A' that contains γ . By replication, we see that $\|(f_n)|_{\gamma_1 \cup \gamma_2}\|_\infty$ is bounded and consequently the Lipschitz norm of f_n on compact subsets of A' is bounded. Up to extracting a subsequence, one may assume that (a suitable interpolation of) (f_n) converges uniformly on compact subsets of A' to a non vanishing (since it has uniform norm 1 on γ) Lipschitz function f . More precisely, we may write $f_n = R_B(g_n) + \overline{R_B(h_n)}$, where (g_n) and (h_n) converge in Lipschitz norm on compact subsets of A' (again by replication). Consequently, $g = \lim g_n$ is in $C_i \cap C_0 = \{0\}$, $h = \lim h_n$ is in $\bar{C}_i \cap \bar{C}_0 = \{0\}$, yielding the needed contradiction.

We may now address the problem of gluing inverting kernels. Assume that $S_g : A^2 \rightarrow \mathbb{C}$ is a kernel with the same regularity conditions as S_i, S_o and such that: for any simple cycle γ in A disconnecting 0 from ∞ , if w is outside γ , $S_g(., w) \in C_i$ and $S_g(., w) - S_o(., w) \in C_o$; and if w is inside γ , $S_g(., w) \in C_o$ and $S_g(., w) - S_i(., w) \in C_i$ (where C_i, C_o are the inner and outer Cauchy data space on γ). Similarly, we assume given $\bar{S}_g : A^2 \rightarrow \mathbb{C}$ a kernel compatible in the same way with inner and outer Cauchy data spaces \bar{C}_i, \bar{C}_o , and with the same regularity as \bar{S}_i, \bar{S}_o . Given this data, we start with constructing \tilde{S}_g , an approximate inverting kernel for $K_g : \mathbb{C}^{\Xi_g^B} \rightarrow \mathbb{C}^{\Xi_g^W}$ (at least for some w 's).

First let us consider two disjoint simple cycles γ_1, γ_2 in A that disconnect 0 from ∞ , with γ_1 inside γ_2 ; γ_i^δ is an approximation of γ_i on Γ_δ . For $w \in M_W^\delta$ within $O(\delta)$ of γ_2 , we set:

$$\begin{aligned} \tilde{S}_g(b, w) &= K_o^{-1}(b, w) + P_o^{\gamma_1^\delta} \left(\frac{1}{2} R_B(e^{i\nu(w)}(S_g(., w) - S_o(., w))) + \frac{1}{2} \overline{R_B} \left(e^{-i\nu(w)}(\bar{S}_g(., w) - \bar{S}_o(., w)) \right) \right) \\ &\quad \text{for } b \text{ outside } \gamma_1 \\ &= P_i^{\gamma_1^\delta} \left(\frac{1}{2} R_B \left(e^{i\nu(w)} S_g(., w) \right) + \frac{1}{2} \overline{R_B} \left(e^{-i\nu(w)} \bar{S}_g(., w) \right) \right) \\ &\quad \text{for } b \text{ inside } \gamma_1 \end{aligned}$$

and for b on γ_1^δ one may use either definition (here $P_i^{\gamma^\delta}$, $P_o^{\gamma^\delta}$ denote the inner and outer discrete Cauchy data space projectors described earlier). Symmetrically, for $w \in M_W^\delta$ within $O(\delta)$ of γ_1 , we set:

$$\begin{aligned}\tilde{S}_g(b, w) &= K_i^{-1}(b, w) + P_i^{\gamma^\delta} \left(\frac{1}{2} R_B(e^{i\nu(w)}(S_g(\cdot, w) - S_i(\cdot, w))) + \frac{1}{2} \overline{R_B} \left(e^{-i\nu(w)}(\bar{S}_g(\cdot, w) - \bar{S}_i(\cdot, w)) \right) \right) \\ &\quad \text{for } b \text{ inside } \gamma_2 \\ &= P_o^{\gamma^\delta} \left(\frac{1}{2} R_B(e^{i\nu(w)} S_g(\cdot, w)) + \frac{1}{2} \overline{R_B} \left(e^{-i\nu(w)} \bar{S}_g(\cdot, w) \right) \right) \\ &\quad \text{for } b \text{ outside } \gamma_2\end{aligned}$$

Let us observe that if w is within $O(\delta)$ of γ_i , $K_g \tilde{S}_g(\cdot, w) - \delta_w$ is supported on white vertices with graph distance ≤ 1 to γ_{3-i}^δ , $i \in \{1, 2\}$; let us denote $\hat{\gamma}_i^W$ these sets of white vertices, $i \in \{1, 2\}$.

In the continuous limit, the assumptions on compatibility of S_g with the Cauchy data spaces specified by S_i , S_o translate into the replication identities:

$$P_i^{\gamma_1}(S_g(\cdot, w)) = S_g(\cdot, w), \quad P_o^{\gamma_1}(S_g(\cdot, w) - S_o(\cdot, w)) = S_g(\cdot, w) - S_o(\cdot, w)$$

on γ_1 if w is on γ_2 ; and symmetrically

$$P_o^{\gamma_2}(S_g(\cdot, w)) = S_g(\cdot, w), \quad P_i^{\gamma_2}(S_g(\cdot, w) - S_i(\cdot, w)) = S_g(\cdot, w) - S_i(\cdot, w)$$

on γ_2 if w is on γ_1 . The corresponding identities for \bar{S}_g also hold. Together with the earlier convergence result for $P_s^{\gamma^\delta}$, $s \in \{i, o\}$, we get for instance that

$$\|P_i^{\gamma_1^\delta}(R_B(S_g(\cdot, w)) - R_B(\bar{S}_g(\cdot, w)))\|_\infty = O(\tilde{\eta}(\delta))$$

(uniform norm on γ_1^B), uniformly in $w \in \gamma_2$.

Consequently, if w is near γ_2 , we have

$$\tilde{S}_g(b, w) = \frac{1}{2} R_B(e^{i\nu(w)} S_g(b, w)) + \frac{1}{2} R_B(e^{-i\nu(w)} \bar{S}_g(b, w)) + O(\tilde{\eta}(\delta))$$

for b a black vertex in a compact subset of $A \setminus \gamma_1$, and in particular for b on either side of γ_1 (ie both definitions of \tilde{S}_g agree up to $O(\tilde{\eta}(\delta))$ near γ_1). It follows that $K_g \tilde{S}_g(\cdot, w) - \delta_w$ is supported on $\hat{\gamma}_1^W$ and is $O(\delta \tilde{\eta}(\delta))$ there. Thus $(K_g \tilde{S}_g)$, seen as an operator $\mathbb{C}^{\hat{\gamma}^W} \rightarrow \mathbb{C}^{\hat{\gamma}^W}$ (where $\hat{\gamma}^W = \hat{\gamma}_1^W \sqcup \hat{\gamma}_2^W$) is such that $\|K_g \tilde{S}_g - \text{Id}\|_{L^1(\gamma^W)} = O(\tilde{\eta}(\delta))$ (since $|\gamma^W| = O(\delta^{-1})$). Thus for δ small enough, $K_g \tilde{S}_g : \mathbb{C}^{\hat{\gamma}^W} \rightarrow \mathbb{C}^{\hat{\gamma}^W}$ is invertible, and $\|(K_g \tilde{S}_g)^{-1} - \text{Id}\|_{L^1(\gamma^W)} = O(\tilde{\eta}(\delta))$. Then for $w \in \gamma^W$, we set

$$S_g^\delta(\cdot, w) = \sum_{w' \in \gamma^W} (K_g \tilde{S}_g)^{-1}(w, w') \tilde{S}_g(\cdot, w')$$

so that $K_g S_g^\delta(\cdot, w) = \delta_w$. It follows that

$$S_g^\delta(b, w) = \frac{1}{2} R_B(e^{i\nu(w)} S_g(b, w)) + \frac{1}{2} R_B(e^{-i\nu(w)} \bar{S}_g(b, w)) + O(\tilde{\eta}(\delta))$$

uniformly in $w \in \gamma^W$ and b in a compact subset of $A \setminus (\gamma_1 \cup \gamma_2)$.

For w in general position in A , w is in the exterior of γ_1 or in the interior of γ_2 (or both). The two cases are similar, so assume that w is in a compact set of $A \setminus \gamma_1$, on the exterior of γ_1 . Then we define $\tilde{S}_g(\cdot, w)$ as we did for w near γ_2 . Then $K_g \tilde{S}_g(\cdot, w) - \delta_w$ is supported on $\hat{\gamma}_1^W$, is $o(\delta)$ there, and we may set:

$$S_g^\delta(\cdot, w) = \tilde{S}_g(\cdot, w) + \sum_{w' \in \hat{\gamma}_1^W} (K_g(\tilde{S}_g(\cdot, w)))(w') S_g^\delta(\cdot, w')$$

so that $K_g S_g^\delta(., w) = \delta_w$. We know that there is a most one f vanishing at infinity such that $K_g f = \delta_w$ (for δ small enough). Thus $S_g^\delta(., w)$ does not depend on the choice of γ_1, γ_2 . By moving γ_1, γ_2 towards the boundary cycles of A , we can extend estimates of $S_g^\delta(b, w)$ for (b, w) in a compact subset of $A^2 \setminus \Delta_A$.

Finally for w outside of A , it is easy to construct $S_g^\delta(., w)$. For instance if w is in $D_o \setminus D_i$, one starts with a truncation of $K_o^{-1}(., w)$ and set

$$S_g^\delta(., w) = \mathbf{1}_{\gamma^{int}} K_o^{-1}(., w) - \sum_{w' \neq w} K_g(\mathbf{1}_{\gamma^{int}} K_o^{-1}(., w)) S_g^\delta(., w')$$

where γ^{int} denotes the inside of the closed cycle γ .

Let us summarise the results of this section. Recall that M is a bipartite graph derived from a rhombi tiling Λ_δ with edge length δ , δ going to zero along some sequence; it is equipped with a linear operator $K : \mathbb{C}^{M_B} \rightarrow \mathbb{C}^{M_W}$. We consider D_i (resp. D_o) a simply connected neighbourhood of 0 (resp. ∞) in $\tilde{\mathbb{C}}$, such that $A = D_i \cap D_o$ is an annulus separating 0 from ∞ . The (sequences of) graphs $\Xi_s = \Xi_s^\delta$, $s \in \{i, o\}$, are bipartite graphs equipped with $K_s : \mathbb{C}^{\Xi_s^B} \rightarrow \mathbb{C}^{\Xi_s^W}$, a nearest neighbour (or finite range) linear operator. The pair (Ξ_i, K_i) is obtained by modifying (M, K) in $D_i \setminus D_o$, and vice versa for (Ξ_o, K_o) . In particular (Ξ_i, K_i) and (Ξ_o, K_o) agree with (M, K) in A . The glued data (Ξ_g, K_g) agrees with (Ξ_s, K_s) in D_s , $s \in \{i, o\}$. We also assume that for any $w \in \Xi_i^W$, there is a unique $K_s^{-1}(., w) \in \mathbb{C}^{\Xi_i^B}$ vanishing at infinity such that $K_s(K_s^{-1}(., w)) = \delta_w$, $s \in \{i, o\}$.

Lemma 8. *Assume that $S_i, \bar{S}_i, S_o, \bar{S}_o : A^2 \setminus \Delta_A \rightarrow \mathbb{C}$ are such that*

$$K_s^{-1}(b, w) = \frac{1}{2} R_B(e^{i\nu(w)} S_s(b, w)) + \frac{1}{2} \overline{R_B}(e^{-i\nu(w)} \bar{S}_s(b, w)) + O(\eta(\delta))$$

uniformly in compact sets of $A^2 \setminus \Delta_A$, $s \in \{i, o\}$, with $(z, w) \mapsto S_i(z, w) - \frac{1}{\pi(z-w)}$ (resp. $\bar{S}_i(z, w) - \frac{1}{\pi(z-w)}$) C^1 in A^2 , and $\lim_{\delta \searrow 0} \eta(\delta) = 0$. Assume that there are $S_g, \bar{S}_g : A^2 \setminus \Delta_A \rightarrow \mathbb{C}$ with same regularity such that: if γ is a fixed simple cycle in A disconnecting 0 from ∞ , then for $w \in A$ outside γ , $S_g(., w)|_\gamma$ is in the inner Cauchy data space C_i defined by S_i and $(S_g(., w) - S_o(., w))|_\gamma$ is in the outer Cauchy data space C_o defined by S_o ; and the corresponding conditions for \bar{S}_g and for $w \in A$ inside γ also hold. Assume that $C_i \cap C_o = \{0\}$, $\bar{C}_i \cap \bar{C}_o = \{0\}$.

Then for δ small enough, for each $w \in \Xi_g^W$ there is a unique $K_g^{-1}(., w) \in \mathbb{C}^{\Xi_g^B}$ vanishing at infinity such that $K_g(K_g^{-1}(., w)) = \delta_w$, and

$$K_g^{-1}(b, w) = \frac{1}{2} R_B(e^{i\nu(w)} S_g(b, w)) + \frac{1}{2} \overline{R_B}(e^{-i\nu(w)} \bar{S}_g(b, w)) + O(\eta(\delta) \vee \delta |\log \delta|)$$

uniformly in compact sets of $A^2 \setminus \Delta_A$, $s \in \{i, o\}$.

Let us remark that in the case where Ξ_o is actually bounded, the “vanishing at ∞ ” condition is void. Conceivably, for unbounded Ξ_o (agreeing with M far enough), one might find use for “divisor” type boundary conditions at infinity: $f(z) = O(|z|^n)$ as $z \rightarrow \infty$, $n \in \mathbb{Z}$ fixed. In this case the previous surgery argument is still valid.

7 Electric correlators

We are now interested in scaling limits of vertex correlators. For instance in the plane, one may consider asymptotics of

$$\langle e^{i \sum_j \alpha_j h(z_j)} \rangle$$

where $\sum \alpha_j = 0$. Heuristically, we expect these asymptotics to be governed by electric correlators for a free field $\langle : \exp(i \sum_j \alpha_j \phi(z_j)) : \rangle$, with $g = \pi^{-1}$ (the value read eg from Corollary 4). However, the discrete height function is at every point deterministic modulo \mathbb{Z} (in the normalisation chosen here). Thus

$$(\alpha_i) \mapsto \langle e^{i \sum_j \alpha_j h(z_j)} \rangle$$

is 2π -periodic in each variable, which is not the case of the scalar free field electric correlators:

$$\langle : \exp(2i\pi \sum_j s_j \phi(z_j)) : \rangle_{\mathbb{C}} \propto \prod_{i < j} |z_i - z_j|^{2s_i s_j}$$

This may be seen as a manifestation of the compactified nature of the height field.

The relevant Cauchy-Riemann operators are those associated to the line bundle L_ρ over the punctured sphere $\Sigma = \hat{\mathbb{C}} \setminus \{z_1, \dots, z_n\}$, where $\rho : \pi(\Sigma) \rightarrow \mathbb{U}$ is a unitary character. The two types of variations we shall consider are the isomonodromic family $(z_1, \dots, z_n) \mapsto L_\rho(z_1, \dots, z_n)$, and Jacobian family $\rho \mapsto L_\rho$.

The analysis relies on a rather precise description of the corresponding discrete operators and their inverting kernels, in particular near the diagonal.

For notational simplicity we take $\delta = 1$.

7.1 Discrete holomorphic functions with monodromy

We proceed with a local study of discrete holomorphic functions and inverting kernels in the presence of a singularity. As before, we consider a rhombus tiling Λ of the plane. Let us mark the midpoint v_0 of an edge of Λ (that is, the center of a face of M). Up to scaling and centering we may assume $\delta = 1$, $v_0 = 0$.

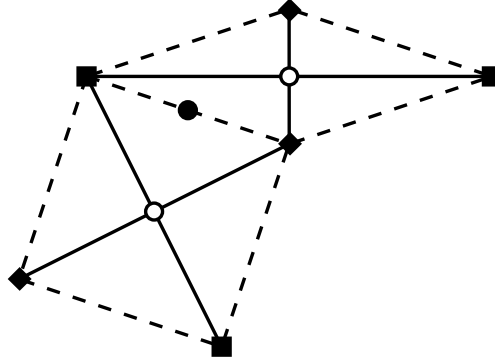


Figure 2: Local geometry of M near the singularity v_0 (black disk)

The unitary characters of $\pi_1(\mathbb{C} \setminus \{v_0\}) \simeq \mathbb{Z}$ are identified with the unit circle \mathbb{U} . Fix a *non trivial* character χ (identified to an element of $\mathbb{U} \neq \{1\}$).

We consider $(\mathbb{C}^{M_B})_\chi$ the functions on the lift of M_B to the universal cover of the punctured plane $\mathbb{C} \setminus \{v_0\}$ which belong to the character χ . Elements of $(\mathbb{C}^{M_B})_\chi$ may also be seen as multiplicatively multivalued elements of \mathbb{C}^{M_B} . The space $(\mathbb{C}^{M_W})_\chi$ is defined similarly; we also consider the restrictions $(\mathbb{C}^{M_V})_\chi$ and $(\mathbb{C}^{M_F})_\chi$. We are concerned with the operators

$$\begin{aligned} \Delta_\Gamma : (\mathbb{C}^{M_V})_\chi &\longrightarrow (\mathbb{C}^{M_V})_\chi \\ \mathbf{K} : (\mathbb{C}^{M_B})_\chi &\longrightarrow (\mathbb{C}^{M_W})_\chi \end{aligned}$$

We begin with a few basic estimates on harmonic functions. In what follows, $B(0, R) = \{z \in M_V : |z| \leq R\}$.

Lemma 9. 1. *There is $\varepsilon = \varepsilon(\chi) > 0$ such that for n large enough, if $f \in (\mathbb{C}^{M_B})_\chi$ is harmonic in $B(0, 2n)$, then*

$$\sup_{x \in B(0, n)} |f(x)| \leq (1 - \varepsilon) \sup_{x \in \partial B(0, 2n)} |f(x)|$$

Similarly, if $f \in (\mathbb{C}^{M_B})_\chi$ is harmonic in $A(n, 3n) = B(0, 3n) \setminus B(0, n)$, then

$$\sup_{x \in A(\frac{3}{2}n, \frac{5}{2}n)} |f(x)| \leq (1 - \varepsilon) \sup_{x \in \partial A(n, 3n)} |f(x)|$$

2. *If $f \in (\mathbb{C}^{M_B})_\chi$ is bounded and harmonic, $f \equiv 0$.*
3. *If $y \in M_V$, there is at most one function $G_\chi(\cdot, y) \in (\mathbb{C}^{M_V})_\chi$ vanishing at infinity such that $\Delta_\Gamma G_\chi(\cdot, y) = \delta_y$ (here δ_y designates an element of $(\mathbb{C}^{M_V})_\chi$ such that $\delta_y(y) \in \{\chi^n : n \in \mathbb{Z}\}$ and $\delta_y(x) = 0$ otherwise).*
4. *If $y \in M_V$, $G_\chi(\cdot, y)$ exists and satisfies*

$$G_\chi(x, y) = O\left(\left|\frac{x}{y}\right|^\varepsilon \wedge \left|\frac{y}{x}\right|^\varepsilon\right)$$

if $|x - y| \geq |y|/2$, for some $\varepsilon = \varepsilon(\chi) > 0$.

5. *For all $x, y \in M_V$, $G_\chi(x, y) = G_{\bar{\chi}}(y, x)$.*
6. *If $y \sim y'$,*

$$G_\chi(x, y') - G_\chi(x, y) = O(|y|^{-1} \left(\left|\frac{x}{y}\right|^\varepsilon \wedge \left|\frac{y}{x}\right|^\varepsilon\right))$$

if $|x - y| \geq |y|/4$ and $G_\chi(x, y') - G_\chi(x, y) = O(|x - y|^{-1})$ otherwise.

Proof. 1. Take $x \in \partial B(0, n)$; up to rotation, we may assume $\arg(x) = 0$. Consider

$$C = A\left(\frac{2}{3}n, \frac{4}{3}n\right) \cap \{z : |\arg(z)| \leq \pi - \varepsilon_0\},$$

$\varepsilon_0 > 0$ small enough and fixed. From the convergence of discrete harmonic measure (see [7], Theorem 3.8), we may deduce that there exists $\eta > 0$ such that for n large enough, the probability that the random walk on Γ started from x exits C on either the top or bottom side of the cone $\{z : |\arg(z)| \leq \pi - \varepsilon_0\}$ is at least η ; let us denote T and B these events. If y is a point on the top side, a random walk starting from y disconnects the bottom side from $\partial B(-x, \frac{n}{2})$ before exiting $B(-x, \frac{n}{2})$ with probability at least $\eta' > 0$ (uniformly in n large enough, $y \in A(\frac{2}{3}n, \frac{4}{3}n)$; this may also be seen using harmonic measure estimates).

Hence with probability at least $\eta' > 0$, we may couple the random walk started from x conditional on T with the random walk conditional on B in such a way that they couple before exiting $B(0, 2n)$ and their winding around 0 differs by 2π . This shows that

$$|f(x)| \leq (\eta\eta'|1 + \chi| + (1 - 2\eta\eta')) \sup_{x \in \partial B(0, 2n)} |f(x)|$$

and since $\chi \neq 1$, we have $|1 + \chi| < 2$. The same argument works in the annular case.

2. By iterating 1., we get

$$|f(x)| \leq (1 - \varepsilon)^n \sup_{x \in \partial B(0, |x|2^n)} |f(x)| \leq (1 - \varepsilon)^n \|f\|_\infty$$

and consequently $f(x) = 0$.

3. Follows from 2.

4. Up to rotation we may assume $\arg(y) = 0$. Let $C_e = \partial B(y, |y|/2)$ and $C_i = \partial B(y, |y|/4)$. For $x \in C_e$, $y \in C_i$, let $\mu_e(y, \{x\})$ be the harmonic measure on C_e seen from $y \in C_i$ and $\mu_i(x, \{y\})$ be the chiral harmonic measure on C_i seen from $x \in C_e$. Explicitly, if C_i^n (resp. y_n) is the lift of C_i (resp. y) to the n -th sheet of the universal cover of $\mathbb{C} \setminus \{0\}$,

$$\mu_i(x, \{y\}) = \sum_{m \in \mathbb{Z}} \chi^m \text{Harm}_{\sqcup C_i^n}(x, \{y_m\})$$

Reasoning as in 1., we can couple a random walk starting from x which approaches the crosscut $(-\infty, 0)$ from above before reaching C_i with a random walk approaching it from below in such a way that, with probability bounded away from 0, if the lift of the first random walk exits at y_m , the second one exits at y_{m-1} . This shows that $\|\mu_i(x, \cdot)\|_{TV} \leq 1 - \varepsilon$ for some constant $\varepsilon > 0$.

If $G_\chi(\cdot, y)$ exists, it satisfies (say for $x \in B(y, |y|/4)$)

$$G_\chi(x, y) = G_{B(y, |y|/2)}(x, y) + \sum_{a \in C_e, b \in C_i} \mu_e(x, \{a\}) \mu_i(a, \{b\}) G_\chi(b, y)$$

Denoting by $T : L^\infty(C_i) \rightarrow L^\infty(C_i)$ the operator:

$$(Tf)(x) = \sum_{a \in C_e, b \in C_i} \mu_e(x, \{a\}) \mu_i(a, \{b\}) f(b)$$

we have $\|T\|_{L^\infty \rightarrow L^\infty} \leq 1 - \varepsilon$ since $\|\mu_i(x, \cdot)\|_{TV} \leq 1 - \varepsilon$ for all x . Conversely, we can set on C_i :

$$G_\chi(\cdot, y) = (\text{Id} - T)^{-1} G_{B(y, |y|/2)}(\cdot, y)|_{C_i}$$

and extend it harmonically outside C_i ; inside C_i we add the harmonic extension to $G_{B(y, |y|/4)}(\cdot, y)$. It is easy to check that the resulting function satisfies the defining condition for G_χ .

We have $G_{B(y, R)}(x, y) = O(1)$ for $x \in B(y, R) \setminus B(y, R/2)$ and $G_{B(y, R)}(x, y) = O(\log |R|)$ for $|x - y| \ll |y|$ (see [7]). Thus $G_\chi(x, y) = O(1)$ for $x \in C_i$. Consequently, by 1. we have

$$G_\chi(x, y) = O\left(\left|\frac{x}{y}\right|^\varepsilon \wedge \left|\frac{y}{x}\right|^\varepsilon\right)$$

if $|x - y| \leq |y|/2$.

5. By uniqueness 3., $G_\chi(x, y)$ is $\bar{\chi}$ -multivalued in y for x fixed (reasoning on the universal cover). To evaluate

$$\Delta_\Gamma^y G_\chi(x, y)$$

we fix y and consider it as a χ -multivalued function in x . Then observe that it decays at infinity and has the same Laplacian as a lifted Dirac mass $\delta_y(\cdot)$. Consequently by 2.,

$$\Delta_\Gamma^y G_\chi(x, y) = \delta_y(x) = \delta_x(y)$$

which concludes.

6. We write $G_\chi(x, y') - G_\chi(x, y) = G_{\bar{\chi}}(y', x) - G_{\bar{\chi}}(y, x)$. If $|x - y| \geq |y|/2$, a combination of the estimate in 4. (for $G_{\bar{\chi}}$) and a discrete Harnack estimate concludes. If $x \in B(y, |y|/4)$, as in 4. we may write

$$G_{\bar{\chi}}(y, x) = G_\Gamma(y, x) - \frac{1}{2\pi} \log |x| + h(y)$$

where G_Γ is the “free Green function” ([7], Theorem 2.5) and h is harmonic in $B(x, |x|/2)$ and uniformly bounded. We conclude with the Harnack estimate and asymptotics for first differences of G_Γ .

□

We now turn to discrete holomorphic functions.

Lemma 10. 1. *The space of χ -multivalued bounded discrete holomorphic functions:*

$$\{f \in (\mathbb{C}^{M_B})_\chi : \|f\|_\infty < \infty, \mathbf{K}f = 0\}$$

is one-dimensional and is spanned by a function f_χ with the following asymptotic expansion:

$$f_\chi(u) = R_B(2^{s-1}\Gamma(1-s)(u-v_0)^{s-1}) + \bar{R}_B(\bar{\tau}2^{-s}\Gamma(s)\overline{(u-v_0)}^{-s}) + O(|u-v_0|^{-s-2} + |u-v_0|^{s-3})$$

where $s \in (0, 1)$, $\chi = e^{2i\pi s}$, and $\tau = (x_0 - v_0)/|x_0 - v_0|$ if x_0 is the vertex of Γ adjacent to the singularity v_0 .

2. *More generally, for $s \in (0, 1)$, $k \in \mathbb{N}$, there exists $f_{k,\chi}$ discrete holomorphic and χ -multivalued s.t.*

$$f_{k,\chi}(u) = \bar{R}_B(\bar{\tau}2^{k-s}\Gamma(s-k)\overline{(u-v_0)}^{k-s}) + O(|u-v_0|^{s-k-1} + |u-v_0|^{k-s-2})$$

with $f_{0,\chi} = f_\chi$ and for b a black vertex adjacent to the singularity,

$$f_{k,\chi}(b) = \frac{2i\pi e^{i\nu(b)}}{(b-v_0)^{k+1-s}} \frac{2^{s-k-1}(-1)^k e^{-i\pi s}}{1 - e^{-2i\pi s}} = (-1)^k e^{i\nu(b)} \frac{\pi}{\sin(\pi s)} \overline{(2(b-v_0))}^{k+1-s}$$

Remark that $\overline{f_{k,\chi}}$ is also discrete holomorphic and χ -multivalued, and that $\overline{f_\chi} = \tau f_\chi$. Let us also point out that if b is a black vertex adjacent to the singularity,

$$f_{k,\chi}(b) = \frac{(-1)^k \pi}{\Gamma(s-k) \sin(\pi s)} \bar{R}_B(\bar{\tau}2^{k-s}\Gamma(s-k)\overline{(b-v_0)}^{k-s})$$

ie plugging b in the leading term of the asymptotic expansion of $f_{k,\chi}$ yields the correct value up to the positive multiplicative constant $\frac{(-1)^k \pi}{\Gamma(s-k) \sin(\pi s)}$.

Proof. 1. Let $f \in (\mathbb{C}^{M_B})$ be bounded with $\mathbf{K}f = 0$. Then a local computation shows that $f|_{M_V}$ is harmonic except at x_0 , the vertex of Γ adjacent to the marked edge of Λ . Consequently, $f|_{M_V}$ decays at infinity and is proportional to $G_\chi(\cdot, x_0)$. Besides $f|_{M_F}$ is locally given as the harmonic conjugate of $f|_{M_F}$ and is thus given modulo a global additive constant (on the lift to the universal cover). This additive constant is determined by the fact that $f|_{M_F}$ is χ -multivalued. Thus the space of bounded harmonic functions in (\mathbb{C}^{M_B}) is at most one-dimensional.

We need to exhibit a non trivial holomorphic function. Adapting the argument for the construction of \mathbf{K}^{-1} in [28] (see also [7], Appendix A, which we follow more closely), we use an integral representation involving the “special” discrete holomorphic functions (or *discrete exponentials*). Recall that the singularity is located at $v_0 = 0$, the midpoint of an edge of Λ which abuts $x_0 \in \Gamma$ and $y_0 \in \Gamma^\dagger$ (thus $x_0 = -y_0$). For a parameter λ , we set

$$e_\lambda(x_0) = \frac{1}{1 - \frac{\lambda}{2}(x_0 - y_0)}$$

$$e_\lambda(y_0) = \frac{1}{1 - \frac{\lambda}{2}(y_0 - x_0)}$$

and for $x \in \Gamma$, $y \in \Gamma^\dagger$ adjacent,

$$\frac{e_\lambda(x)}{e_\lambda(y)} = \frac{1 - \frac{\lambda}{2}(y - x)}{1 - \frac{\lambda}{2}(x - y)}$$

It may be checked that $e_\lambda \in (\mathbb{C}^{M_B})$ is well-defined and discrete holomorphic in the sense that $Ke_\lambda = 0$ (recall that K is obtained from K by a gauge change). For $u \in M_B$, we have $e_\lambda(u) = O(1/|\lambda|)$ as $\lambda \rightarrow \infty$. Besides, one can choose a path $(u_0 \dots u_n)$ on Λ from $u_0 = x_0$ or y_0 to $u_n = u$ in such a way that $u_0, u_1 - u_0, \dots, u_n - u_{n-1}$ lie in the cone:

$$\{z : |\arg(z) - \arg(u)| \leq \pi - \varepsilon_0\}$$

for some $\varepsilon_0 > 0$. Consequently the poles of $e_\lambda(u)$ are in $\{\lambda : |\arg(\lambda) - \arg(\bar{u})| \leq \pi - \varepsilon_0\}$ (and have norm $\delta/2$).

For $s \in (0, 1)$, set

$$f_s(u) = \int_0^{\bar{u}\infty} \lambda^{-s} e_{-\lambda}(u) d\lambda$$

which is $\chi = e^{2i\pi s}$ -multivalued. Here $\int_0^{\bar{u}\infty}$ represents integration along a half-line from 0 to infinity in the direction \bar{u} . If $w \in M_W$, u_1, \dots, u_4 its four black neighbours, one can show that the same integration path may be used for u_1, \dots, u_4 and consequently by linearity $(Kf_s)(w) = 0$.

For small λ we have

$$e_\lambda(u) = \exp(\lambda u + O(|u\lambda^3| + |\lambda^2|))$$

and for large λ

$$e_\lambda(u) = -\frac{\epsilon}{\lambda(x_0 - v_0)} \exp(4\frac{\bar{u}}{\lambda} + O(|u\lambda^{-3}| + |\lambda^{-2}|))$$

where $\epsilon = 1$ on Γ and $\epsilon = -1$ on Γ^\dagger (as in [35]). (Notice that $\frac{1+\epsilon}{1-\epsilon} = \exp(2\epsilon + O(\epsilon^3))$). The asymptotic expansion comes from small values of λ (say $|\lambda| \leq 1/\sqrt{|u|}$) and large values (say $|\lambda| \geq \sqrt{|u|}$). The intermediate values of λ contribute to an exponentially small error in the asymptotic analysis. We have

$$\begin{aligned} \int_0^{\bar{u}\infty} \lambda^{-s} \exp(-\lambda u) d\lambda &= \Gamma(1-s) u^{s-1} \\ \int_0^{\bar{u}\infty} \lambda^{-s} \left(\frac{\epsilon}{\lambda(x_0 - v_0)} \exp(-4\bar{u}\lambda^{-1}) \right) d\lambda &= \frac{\epsilon}{x_0 - v_0} \Gamma(s) (4\bar{u})^{-s} \end{aligned}$$

The error terms are estimated via

$$\int_0^{1/\sqrt{|u|}} t^{-s} \exp(-|u|t) O(|u|t^3 + t^2) dt = \int_0^{\sqrt{|u|}} t^{-s} \exp(-t) O(t^3 + t^2) \frac{dt}{|u|^{3-s}} = O(|u|^{s-3})$$

and similarly for the other integral. This shows in particular that f_s is not identically zero. We then obtain f_χ by multiplying by a constant and applying a gauge transformation.

2. For $k \in \mathbb{N}$, $s \in (0, 1)$, set

$$f_{k,s}(u) = \int_\gamma \lambda^{k-s} e_{-\lambda}(u) d\lambda$$

where $\gamma = \gamma(u)$ is an integration contour consisting of a large counterclockwise circle around zero (encircling all poles of the $e_\cdot(u)$'s); two connecting segments on $\bar{u}(0, \infty)$; and a small clockwise circle around 0 (with all poles on its exterior). As before it may be checked that $Kf_{k,s} = 0$, since for a given $w \in M_W$, the same integration contour may be used for all black neighbours of w .

We choose a lift of the mapping $u \mapsto \lambda = -u^{-1}$ to the covers of $\{u : u \neq v_0\}$ and $\{\lambda : \lambda \neq 0\}$. We choose determinations of \log on these pointed covers in such a way that for $\lambda = u^{-1}$, $\log(\lambda) = -\log(u)$. Then $\gamma(u)$ is a simple contour on the cover of $\{\lambda : \lambda \neq 0\}$, for instance by specifying that the outward segment is in the direction u^{-1} ; then the inward segment is on the next sheet of the cover.

Notice that, up to rotating the lattice around v_0 , one may assume that $u - v_0 \in (0, \infty)$, in which case one may a fixed branch of \log , which will be convenient for local computations. More precisely, if $M' = e^{i\theta} M$, one may choose branches in the definitions of $f_{k,s}^M, f_{k,s}^{M'}$ in such a way that $f_{k,s}^{M'}(e^{i\theta} u) = e^{is\theta} f_{k,s}^M(u)$.

In order to obtain an asymptotic expansion for large $R = |u|$, one may take the outer circle of the integration contour with radius R and the inner circle with radius R^{-1} . The contribution of the inner half of the contour γ is, modulo exponentially small error terms,

$$R^{s-k-1} \int_{\gamma_i} \lambda^{k-s} \exp(-\lambda u/R + O(u\lambda^2/R^2)) d\lambda = O(|u|^{s-k-1})$$

where γ_i is a contour consisting of a clockwise unit circle around 0 connected to infinity by two half-lines along $\bar{u}(0, \infty)$. Similarly, the contribution of the outer half of γ is:

$$R^{(k-s)} \int_{\gamma_o} \lambda^{k-s} \left(\frac{\epsilon}{\lambda(x_0 - v_0)} \exp(-4\bar{u}(R\lambda)^{-1} + O(u(R\lambda)^{-3} + (R\lambda)^{-2}) \right) d\lambda$$

where γ_o consists of the unit circle connected to zero by two rays along $\bar{u}(0, \infty)$.

Up to rotation and inversion, we have to evaluate

$$\int_{\gamma_0} t^{s'-1} e^{-t} dt$$

where γ_0 is a contour consisting of a clockwise circle around 0 connected to infinity by two half-lines along $(0, \infty)$. (For definiteness, set $\log(re^{i\theta}) = \log r + i\theta$ for $\theta \in [0, 2\pi)$, and accordingly $t^{s'-1} = \exp((s'-1)\log(t))$). We observe that, by contour deformation,

$$\int_{\gamma_0} t^{s'-1} e^{-t} dt = (1 - e^{2i\pi s'}) \Gamma(s')$$

when $s' \in (0, \infty)$; and that both sides are analytic in s' (the RHS has removable singularities on $-\mathbb{N}$). Consequently they agree for all $s' \in \mathbb{C}$. Thus

$$f_{k,s}(u) = \epsilon \frac{(1 - \chi^{-1})}{x_0 - v_0} \Gamma(s - k) (4\bar{u})^{k-s} + O(|u|^{k-s-2} + |u|^{s-k-1})$$

On the other hand, for $u \in \{x_0, y_0\}$, the residue formula yields:

$$f_{k,s}(u) = \int_{\gamma} \lambda^{k-s} \frac{1}{1 + \lambda(u - v_0)} d\lambda = 2i\pi e^{i\pi(k-s)} (u - v_0)^{s-k-1}$$

□

Lemma 11. *Let $w \in M_W$ be one of the two white vertices adjacent to the singularity v_0 . Then there is $g_w \in (\mathbb{C}^{M_B})_{\chi}$ which is discrete holomorphic outside of w , has asymptotic expansion*

$$g_w(u) = R_B \left(2^s e^{i\nu(w)} \Gamma(1-s) (u - v_0)^{s-1} \right) + \bar{R}_B \left(2^{-s} e^{-i\nu(w)} \Gamma(1+s) \overline{u - v_0}^{-s-1} \right) + O(|u|^{-s-3} + |u|^{s-3})$$

and such that

$$(\mathbf{K}g_w)(w) = 2^{1+s} \pi (w - v_0)^s$$

Remark that for $s = 0$, $g_w = 2\pi \underline{\mathbf{K}}^{-1}(\cdot, w)$.

Proof. We proceed as in the construction of f_χ , simply changing the normalisation of the discrete exponentials (or rather reverting to the normalisation used in [28], [7]; we follow closely Appendix A of [7] here). Let $x_0 \in \Gamma$, $y_0 \in \Gamma^\dagger$ be the two black vertices adjacent to the singularity; let $x'_0 \in \Gamma$, $y'_0 \in \Gamma^\dagger$ be such that w corresponds to the edges $(x_0 x'_0)$ and $(y_0 y'_0)$. For $\lambda \in \mathbb{C}$, set

$$e_\lambda(x_0) = \frac{1}{(1 - \frac{\lambda}{2}(x_0 - y_0))(1 - \frac{\lambda}{2}(x_0 - y'_0))}$$

and for $x \in \Gamma$, $y \in \Gamma^\dagger$ adjacent,

$$\frac{e_\lambda(x)}{e_\lambda(y)} = \frac{1 - \frac{\lambda}{2}(y - x)}{1 - \frac{\lambda}{2}(x - y)}$$

Then $Ke_\lambda = 0$ and we have the following estimates:

$$\begin{aligned} e_\lambda(u) &= \exp(\lambda(u - w) + O(|u\lambda^3| + |\lambda^2|)) \\ e_\lambda(u) &= \frac{4\epsilon}{(x_0 - y_0)(x_0 - y'_0)\lambda^2} \exp(4\frac{\overline{u - w}}{\lambda} + O(|u\lambda^{-3}| + |\lambda^{-2}|)) \end{aligned}$$

For notational simplicity, we now assume that $w = 0$.

Fix a ray $\bar{u}_0(0, \infty)$ from $w = 0$ to infinity which is disjoint of the rays $\bar{u}(0, \infty)$ for all $u \in M$, and such that $\arg(u_0 - v_0) \in (-\pi, \pi)$. We also fix a branch of \log in $\mathbb{C} \setminus \bar{u}_0[0, \infty)$ and set $z^s = \exp(s \log(z))$ there. By rotating the lattice we may assume that $u_0 = -1$ and \log is the usual branch in $\mathbb{C} \setminus (-\infty, 0]$ (so that $(z^{-1})^s = z^{-s}$ outside of the branch cut).

Then set

$$\tilde{g}(u) = \int_{-\bar{u}\infty}^0 (-\lambda)^{-s} e_\lambda(u) d\lambda$$

which defines a single valued function on M_B . It can be identified with an element of $(\mathbb{C}^{M_B})_\chi$ by taking $u_0(0, \infty)$ as a branch cut. As before, one then checks that $K\tilde{g}$ vanishes except at w , by deformation of the integration path.

We have

$$\begin{aligned} \int_{-\bar{u}\infty}^0 (-\lambda)^{-s} \exp(\lambda u) d\lambda &= \Gamma(1 - s) u^{s-1} \\ \int_{-\bar{u}\infty}^0 (-\lambda)^{-s} \left(\frac{4\epsilon}{(x_0 - y_0)(x_0 - y'_0)\lambda^2} \exp(4\bar{u}\lambda^{-1}) \right) d\lambda &= \frac{4\epsilon}{(x_0 - y_0)(x_0 - y'_0)} \Gamma(1 + s) (4\bar{u})^{-s-1} \end{aligned}$$

Observe that $(y_0 - x_0)(y'_0 - x_0) = e^{2i\nu(w)}$ (if $\delta = 1$). We need to evaluate \tilde{g} at neighbors of w . In order to evaluate $\int_{-\bar{u}\infty}^0 (-\lambda)^{-s} f(\lambda) d\lambda$, where f is rational of degree -2 without poles on the integration path, we may write (here $\chi = e^{2i\pi s}$)

$$(\chi^{-1} - 1) \int_{-\bar{u}\infty}^0 (-\lambda)^{-s} f(\lambda) d\lambda = \oint_\gamma \widetilde{(-\lambda)^{-s}} f(\lambda) d\lambda$$

where γ is a closed counterclockwise contour containing all poles of f and not crossing the ray $-\bar{u}(0, \infty)$ and $\lambda \mapsto \widetilde{(-\lambda)^{-s}}$ is the determination of $\lambda \mapsto (-\lambda)^{-s}$ with a branch cut on $-\bar{u}(0, \infty)$ which agrees with the reference determination on the right handside of $-\bar{u}(\infty, 0)$. Specifying to $f(\lambda) = \frac{1}{(1 - \lambda z_1)(1 - \lambda z_2)}$, the residue formula yields

$$\int_{-\bar{u}\infty}^0 (-\lambda)^s \frac{1}{(1 - \lambda z_1)(1 - \lambda z_2)} d\lambda = \frac{2i\pi}{\chi^{-1} - 1} \sum_z \widetilde{(-z)^{-s}} \text{Res}_z(f) = \frac{2i\pi}{(1 - \chi^{-1})(z_2 - z_1)} \left(\widetilde{(-z_2^{-1})^{-s}} - \widetilde{(-z_1^{-1})^{-s}} \right)$$

Let us denote v_0, v_1, v_2, v_3 the midpoints of edges of the rhombus of Λ corresponding to w (where v_0 is the midpoint of $(x_0 y_0)$) listed counterclockwise, and also $(b_0, b_1, b_2, b_3) = (x_0, y_0, x'_0, y'_0)$. We have

$$e_\lambda(b_i) = \frac{1}{(1 - \lambda v_{i-1})(1 - \lambda v_i)}$$

for $i = 0, \dots, 3$, with cyclical indexing. For definiteness, let us assume that u_0 was chosen in the interior of the cone generated by x_0 and v_0 . Then we obtain (in terms of the reference determination)

$$\begin{aligned}\tilde{g}(b_0) &= \frac{2i\pi}{(1 - \chi^{-1})(v_0 - v_3)} ((-v_0)^s - (-v_3)^s) \\ \tilde{g}(b_1) &= \frac{2i\pi}{(1 - \chi^{-1})(v_1 - v_0)} (\chi^{-1}(-v_1)^s - \chi^{-1}(-v_0)^s) \\ \tilde{g}(b_2) &= \frac{2i\pi}{(1 - \chi^{-1})(v_2 - v_1)} ((-v_2)^s - \chi^{-1}(-v_1)^s) \\ \tilde{g}(b_3) &= \frac{2i\pi}{(1 - \chi^{-1})(v_3 - v_2)} ((-v_3)^s - (-v_2)^s)\end{aligned}$$

Taking into account $K(w, b_j) = i(v_{j-1} - v_j)$, we get

$$(K\tilde{g})(w) = 2\pi(-v_0)^s$$

In order to obtain g_w we multiply \tilde{g} by $2^s e^{i\nu(w)}$ and apply a gauge transformation. \square

Lemma 12. *Let $\chi = e^{2i\pi s}$, $s \in (0, \frac{1}{2})$. Let $f \in (\mathbb{C}^{M_B})_\chi$ be bounded and s.t. $\mathbb{K}f = 0$ outside of $B(0, R)$. Then there exists $c = c(f)$ such that for $\varepsilon_0 > 0$, $|z| \geq R$,*

$$f(z) = cf_\chi(z) + O(\|f\|_{B(0, R)} \|z/R\|^{s-1+\varepsilon_0})$$

More precisely, for $s \in (0, \frac{1}{2}]$, there exists $c_1(f) = O(\|f\|_{B(0, R)} R^s)$, $c_2(f) = O(\|f\|_{B(0, R)} R^{1-s})$ such that for $\varepsilon_0 > 0$, $|z| \geq R$,

$$\begin{aligned}f(z) &= c_1 f_\chi(z) + O(\|f\|_{B(0, R)} \|z/R\|^{s-1}) \\ &= c_1 f_\chi(z) + c_2 g_\chi(z) + O(\|f\|_{B(0, R)} \|z/R\|^{-s-1+\varepsilon_0})\end{aligned}$$

where $g_\chi = g_w$ for a $w \in M_W$ adjacent to v_0 .

Proof. Fix $\varepsilon, \eta > 0$. Then there exists $R_0 > 0$ such that for all g discrete harmonic outside of $B(0, R_0)$, there exists a continuous harmonic, χ -multivalued function \hat{g} s.t. for all $z \in M_V$, $|z| \geq (1 + \eta)R_0$

$$|g - \hat{g}|(z) \leq \varepsilon \|g|_{\partial B(0, R_0)}\|_\infty$$

Moreover R_0 is uniform in the choice of rhombus tiling Λ under (3.1). This is obtained by contradiction using equicontinuity and Ascoli-Arzelà (see [7] for similar arguments). Similarly, if g is discrete holomorphic, then $g|_{M_F}$ converges to a continuous harmonic function \hat{g}^\dagger , which is conjugate to \hat{g} .

Let \hat{g} be bounded, harmonic, χ -multivalued on $\{z : |z| > 1\}$. Then \hat{g} can be written locally as the sum of an holomorphic function h and an antiholomorphic function \tilde{h} . The decomposition $\hat{g} = h + \tilde{h}$ is unique up to additive constants. It is easy to see that the additive constant can be set so that h, \tilde{h} are χ -multivalued. From the Laurent expansion of $h(z)z^{-s}$, $\tilde{h}(z)\bar{z}^{s-1}$, we get that

$$\hat{g}(z) = \sum_{n \geq 1} a_n z^{s-n} + \sum_{n \geq 0} b_n \bar{z}^{-s-n}$$

for coefficients $(a_n), (b_n)$ which are bounded by $\|\hat{g}\|_{\partial B(0,1)}\|_\infty$. Thus for all $|z| \geq C > 1$,

$$|\hat{g}(z) - b_0 \bar{z}^{-s}| \leq |z|^{s-1} \frac{C(1+C^{-2s})}{C-1} \|\hat{g}\|_{\partial B(0,1)}\|_\infty$$

Let us now consider f (restricted to M_V for notational simplicity). Let \hat{g}_n be continuous, harmonic such that for $|z| \geq (1+\eta)R_n$

$$|f - \hat{g}_n|(z) \leq \varepsilon \|f\|_{\partial B(0,R_n)}\|_\infty$$

(ε, R_n to be fixed later). There is $c > 0$, $\alpha_n = O(\|g\|_{\partial B(0,R_n)}\|_\infty R^s)$ such that for $C > 1$ large enough, $|z| \geq CR_n$

$$|\hat{g}_n(z) - \alpha_n \bar{z}^{-s}| \leq |z|^{s-1} (1 + cC^{-1}) \|\hat{g}\|_{\partial B(0,(1+\eta)R_n)}\|_\infty \leq |z/R_n|^{s-1} (1 + cC^{-1})(1 + \varepsilon) \|f\|_{\partial B(0,R_n)}\|_\infty$$

Given that $f_\chi(z) = \bar{R}_B(\bar{z}^{-s}) + O(|z|^{s-1})$ (up to normalization), we get that for $z \geq CR_n$,

$$|f - \alpha_n f_\chi|(z) \leq (\varepsilon + |z/R_n|^{s-1} (1 + cC^{-1})(1 + \varepsilon)(1 + cR_n^{2s-1})) \|f\|_{\partial B(0,R_n)}\|_\infty$$

Similarly, one can find coefficients β_n such that

$$|f - \beta_{n+1} f_\chi|(z) \leq (\varepsilon + |z/R_n|^{s-1} (1 + \varepsilon')) \|(f - \beta_n f_\chi)|_{\partial B(0,R_n)}\|_\infty$$

for $|z| \geq R_{n+1} = CR_n$. Choose ε small enough and C, R_0 large enough so that $\varepsilon + C^{s-1}(1 + \varepsilon') \leq C^{s-1+\varepsilon_0}$ (and $R_0 \geq R$ in any case). It follows that

$$\|(f - \beta_n f_\chi)|_{\partial B(0,R_n)}\|_\infty = O(\|(f)|_{\partial B(0,R_0)}\|_\infty (R_n/R_0)^{s-1+\varepsilon_0})$$

Since $\|(f - \beta_n f_\chi)|_{\partial B(0,R_{n+1})}\|_\infty$ is of the same order, we get

$$\beta_{n+1} - \beta_n = O(\|(f)|_{\partial B(0,R_0)}\|_\infty (R_n/R_0)^{s-1+\varepsilon_0} R_n^s)$$

If we fix $\varepsilon_0 > 0$ small enough such that $2s - 1 + \varepsilon_0 < 0$, this is summable. Moreover, if $\beta = \lim \beta_n$, we have

$$\beta_n = \beta + O(\|(f)|_{\partial B(0,R_0)}\|_\infty (R_n/R_0)^{s-1+\varepsilon_0} R_n^s)$$

and finally

$$\|(f - \beta f_\chi)|_{\partial B(0,R_n)}\|_\infty = O(\|(f)|_{\partial B(0,R_0)}\|_\infty (R_n/R_0)^{s-1+\varepsilon_0})$$

In order to obtain the more precise statement, we write $\hat{g}(z) = b_0 \bar{z}^{-s} + a_1 z^{s-1} + O(|z|^{-s-1})$ we proceed similarly to show the existence of coefficients (β_n, γ_n) such that

$$\|(f - \beta_n f_\chi - \gamma_n g_\chi)|_{\partial B(0,R_n)}\|_\infty = O(\|(f)|_{\partial B(0,R_0)}\|_\infty (R_n/R_0)^{-s-1+\varepsilon_0})$$

Thus

$$(\beta_{n+1} - \beta_n) f_\chi + (\gamma_{n+1} - \gamma_n) g_\chi = O(\|(f)|_{\partial B(0,R_0)}\|_\infty (R_n/R_0)^{-s-1+\varepsilon_0})$$

on $B(0, R_{n+1})^c$. This shows (by specialising at points close to the positive and negative half-lines respectively, say) that

$$\beta_{n+1} - \beta_n = O(\|(f)|_{\partial B(0,R_0)}\|_\infty (R_n/R_0)^{-s-1+\varepsilon_0} R_n^s)$$

and subsequently

$$\gamma_{n+1} - \gamma_n = O(\|(f)|_{\partial B(0,R_0)}\|_\infty (R_n/R_0)^{-s-1+\varepsilon_0} R_n^{1-s})$$

and one concludes as before. \square

In what follows, we assume without loss of generality that $s \in (0, \frac{1}{2})$. If $s \in (-\frac{1}{2}, 0)$, set $\underline{K}_\chi^{-1} = \overline{\underline{K}_\chi^{-1}}$.

Lemma 13 (Chiral Cauchy kernel). *Let $\chi = e^{2i\pi s}$, $s \in (0, \frac{1}{2})$. For $w \in M_W$, there is a unique function $\underline{K}_\chi^{-1}(\cdot, w) \in (\mathbb{C}^{M_B})_\chi$ such that $\underline{K}\underline{K}_\chi^{-1}(\cdot, w) = \delta_w$ and*

$$\underline{K}_\chi^{-1}(b, w) = O\left(\frac{1}{|w|}(|b|/|w|)^{s-1}\right)$$

for $\varepsilon_0 > 0$, $|b| \geq 2|w|$. Moreover, $\underline{K}_\chi^{-1}(b, w) = O\left(\frac{1}{|w|}(|w|/|b|)^s\right)$ for $|b| \leq |w|/2$.

Proof. The uniqueness follows from the fact that there is no nonzero holomorphic function f s.t. $f(z) = o(|z|^{-s})$.

If w corresponds to the oriented edge $(x_0 x'_0)$ of Γ , we may set

$$S(b) = G_\chi(b, x'_0) - G_\chi(b, x_0)$$

for $b \in \Gamma$. One may extend S to Γ^\dagger in such a way that $S(y_0) = 0$ (y_0 is the vertex of Γ^\dagger adjacent to the vertex singularity at $v_0 = 0$) and $\underline{K}S = 0$ except at w . We know that

$$S(b) = O\left(\frac{1}{|w|}((|b|/|w|)^\varepsilon \wedge (|w|/|b|)^\varepsilon)\right)$$

for $b \in \Gamma$, $|b - w| \geq |w|/4$.

Consider a branch cut running from 0 to infinity in the half-space $\{z : \Re(z\bar{w}) \leq 0\}$. Applying the discrete Green's formula in the slit plane (first intersecting it with a large disc) shows that the sum of (weighted) discrete derivatives of S across the slit vanishes. By harmonic conjugation, this is saying that $S(b) \rightarrow 0$ as $b \rightarrow \infty$ on Γ^\dagger .

By harmonic conjugation and the Harnack principle, we get the following estimates on S for $|b - w| \geq |w|/4$, $b \in \Gamma^\dagger$, by integrating along a ray from y_0 to b if $|b| \leq |w|$ and from ∞ to $|b|$ if $|b| \geq |w|$:

$$S(b) = O\left(\frac{\varepsilon^{-1}}{|w|}((|b|/|w|)^\varepsilon \wedge (|w|/|b|)^\varepsilon)\right)$$

By the previous lemma, there exists $\beta = O(\|f\|_{\partial B(0, 2|w|)}|w|^s)$ such that $\underline{K}_\chi(\cdot, w) = S(\cdot) - \beta f_\chi$ satisfies

$$\underline{K}_\chi(b, w) = O(|w|^{-1}(|b|/|w|)^{s-1})$$

for $|b| \geq 2|w|$, say, and $\underline{K}_\chi(b, w) = O(|w|^{-1}(|w|/|b|)^s)$ for $|b| \leq |w|$, $|b - w| \geq |w|/4$. □

Lemma 14. *Let $\chi = e^{2i\pi s}$, $s \in (0, \frac{1}{2})$. Let $f \in (\mathbb{C}^{M_B})_\chi$ be bounded and s.t. $\underline{K}f = 0$ in $B(0, R)$. Then*

$$f(z) = O(\|f\|_{\partial B(0, R)}|z/R|^{-s})$$

Proof. Let $\tilde{f} = f\mathbf{1}_{B(0, R)}$. Then $\underline{K}\tilde{f} = O(\|f\|_{\partial B(0, R)})$ and the support of $\underline{K}\tilde{f}$ is contained in white vertices adjacent to black vertices belonging to $\partial B(0, R) \subset M_B$; let us use the same notation for this subset of M_W . Then consider

$$\tilde{f} - \sum_{w \in \partial B(0, R)} (\underline{K}\tilde{f})(w) \underline{K}_\chi(w, \cdot)$$

This function in $(\mathbb{C}^{M_B})_\chi$ is discrete holomorphic and $O(|z|^{s-1})$ at infinity, hence vanishes identically. Consequently, we have the (chiral) Cauchy integral formula:

$$f(b) = \sum_{w \in \partial B(0, R)} (\underline{K}\tilde{f})(w) \underline{K}_\chi(w, b) = O(\|f\|_{\partial B(0, R)}|b/R|^{-s})$$

in $B(0, R)$. □

Lemma 15. Let $\chi = e^{2i\pi s}$, $s \in (0, \frac{1}{2})$.

1. The kernel \underline{K}_χ^{-1} satisfies:

$$\underline{K}_\chi^{-1}(b, w) = \frac{1}{2}R_B \left(\frac{e^{i\nu(w)}}{\pi(b-w)} \left(\frac{b}{w} \right)^s \right) + \frac{1}{2}\overline{R_B} \left(\frac{e^{-i\nu(w)}}{\pi(\overline{b-w})} \left(\frac{\bar{b}}{\bar{w}} \right)^{-s} \right) + O(|w|^{2s-2})$$

for $\frac{1}{2}|w| \leq |b| \leq 2|w|$, $|b-w| \geq \frac{1}{4}|w|$.

2. If w is adjacent to the singularity $v_0 = 0$, we have

$$\underline{K}_\chi^{-1}(b, w) = \frac{\Gamma(1-s)}{2}R_B \left(\frac{e^{i\nu(w)}}{\pi(b-w)} \left(\frac{b}{w} \right)^s \right) + \frac{\Gamma(1+s)}{2}\overline{R_B} \left(\frac{e^{-i\nu(w)}}{\pi(\overline{b-w})} \left(\frac{\bar{b}}{\bar{w}} \right)^{-s} \right) + O(|b|^{s-3})$$

and if moreover the edge (bw) is adjacent to the singularity, and v_1 is the symmetric image of v_0 across (bw) , then for $b \in \Gamma$, v_1, b, b_0 in counterclockwise order around w ,

$$\mathbf{K}(w, b)\underline{K}_\chi^{-1}(b, w) = \frac{1}{1-\chi^{-1}} \left(1 - \left(\frac{w-v_1}{w-v_0} \right)^s \right)$$

Note that the conditions in 2. are not restrictive, by exchanging the roles of Γ, Γ^\dagger and/or conjugating the lattice. Also, remark that

$$\lim_{s \searrow 0} \frac{1 - e^{is(\arg(w-v_1) - \arg(w-v_0))}}{1 - e^{-2i\pi s}} = \frac{1}{2\pi}(\arg(w-v_0) - \arg(w-v_1)) = \mathbf{K}(w, b)\underline{K}^{-1}(b, w)$$

Before proceeding with the proof, let us sketch an alternative construction for $\underline{K}_\chi^{-1}(\cdot, w)$, for w adjacent to v_0 , from which one can deduce the evaluation $\underline{K}_\chi^{-1}(b, w)$ for b, w adjacent to v_0 . Consider a branch cut γ (a simple path on M^\dagger) running from infinity to v_0 and v_1 : $\gamma = (\dots, v_{-2}, v_{-1}, v_0, v_1)$. We have constructed basic discrete holomorphic functions f_{χ, v_i} with monodromy at v_i , $i = 0, 1$. Using the branch cut γ , we can look at them as single valued on the same fundamental domain. Then f_{χ, v_1} , seen as a function with singularity at v_0 , is discrete holomorphic except at w . From the asymptotic expansions, we can find a (non trivial) linear combination of f_{χ, v_0} , f_{χ, v_1} which is $o(|z|^{-s})$ as $z \rightarrow \infty$. Thus, by the characterisation of $\underline{K}_\chi^{-1}(\cdot, w)$, it can be written as a linear combination of f_{χ, v_0} , f_{χ, v_1} . From the exact evaluation of these functions near their singularity, one recovers 2.

Proof. 1. Let us fix $w \in M_W$ and set $R = |w|$. We first construct an approximate kernel $\tilde{S} \in (\mathbb{C}^{M_B})_\chi$. If $b \in B(0, r_0)$, set $\tilde{S}(b) = cf_\chi(b)$. If $b \in B(w, r_1)$, set

$$\tilde{S}(b) = \underline{K}^{-1}(b, w) + \frac{1}{2}R_B \left(\frac{e^{i\nu(w)}}{\pi} \cdot \frac{s}{w} \right) + \frac{1}{2}\overline{R_B} \left(\frac{e^{-i\nu(w)}}{\pi} \cdot \frac{-s}{\bar{w}} \right)$$

and otherwise

$$\tilde{S}(b) = \frac{1}{2}R_B \left(\frac{e^{i\nu(w)}}{\pi(b-w)} \left(\frac{b}{w} \right)^s \right) + \frac{1}{2}\overline{R_B} \left(\frac{e^{-i\nu(w)}}{\pi(\overline{b-w})} \left(\frac{\bar{b}}{\bar{w}} \right)^{-s} \right)$$

Here c, r_0, r_1 are parameters yet to be specified. Then $\mathbf{K}\tilde{S}(\cdot, w) = \delta_w$ in $B(0, r_0) \sqcup B(w, r_1)$. Set c s.t.

$$c\tau 2^{-s}\Gamma(s) = -\frac{e^{-i\nu(w)}}{2\pi\bar{w}^{1-s}}$$

Then the discontinuity of \tilde{S} on $\partial B(0, r_0)$ is of order $O(r_0^s/|w|^{s+1} + r_0^{s-1}/|w|^{1-s})$.

The discontinuity of \tilde{S} on $\partial B(w, r_1)$ is of order $O(r_1/|w|^2)$. Elsewhere $\mathbf{K}\tilde{S}(\cdot, w)$ is controlled by the third derivative of the continuous limit.

Next we use the a priori estimate on the kernel \underline{K}_χ^{-1} to correct \tilde{S} . First let us observe that $\underline{K}_\chi^{-1}(w, \cdot) = \tilde{S} - \underline{K}_\chi^{-1}(\underline{K}\tilde{S})$. For this need only check that $(\underline{K}_\chi^{-1}(\underline{K}\tilde{S}))(b) = O(|b|^{s-1})$ as $b \rightarrow \infty$. The contribution from $(\underline{K}\tilde{S})$ in $B(0, 2R)$ is easily seen to be $O(|b|^{s-1})$. The other terms are handled as below.

Let us estimate the correction $\underline{K}_\chi^{-1}(\underline{K}\tilde{S})$ for $b \in A(R/2, 2R)$, $b \notin B(w, R/2)$ (where $R = |w|$, say). We simply add upper bounds (up to multiplicative constants) stemming from $(\underline{K}\tilde{S})(w')$ for respectively: $w' \in B(b, R/4)$; $w' \in \partial B(w, r_1)$; $w' - w \in A(r_1, R/4)$; $w' \in A(R/2, 2R) \setminus (B(b, R/4) \cup B(w, R/4))$; $w' \notin B(0, 2R)$; $w' \in \partial B(0, r_0)$; and $w' \in A(r_0, R/2)$:

$$\sum_{k=1}^R \frac{k}{kR^4} + \frac{r_1}{R^2} \cdot \frac{r_1}{R} + \sum_{k=r_1}^R \frac{k}{k^4} \cdot \frac{1}{R} + \frac{R^2}{R^4 \cdot R} + \sum_{k=2R}^\infty k \cdot \frac{k^{s-4}}{R^s} \cdot \frac{k^s}{kR^s} + r_0 \left(\frac{r_0^s}{R^{s+1}} + (r_0 R)^{s-1} \right) \frac{R^{s-1}}{r_0^s}$$

Thus, up to a constant, we get the upper bound

$$\frac{r_1^2}{R^3} + \frac{1}{r_1^2 R} + \frac{r_0}{R^2} + R^{2s-2} = O(R^{2s-2})$$

with $r_1 = O(\sqrt{R})$, $r_0 = O(1)$.

2. Here $\underline{K}_\chi^{-1}(\cdot, w)$ is proportional to g_w . Since $\delta = 1$, we have $(v_0 - w)^s \overline{(v_0 - w)}^s = 2^{-2s}$.

□

We are now in position to state the analogue of Lemma 12 (concerning the asymptotic expansion of a discrete χ -multivalued holomorphic function in a neighbourhood of ∞) for the singularity at zero.

Lemma 16. *Let $\chi = e^{2i\pi s}$, $s \in (0, \frac{1}{2})$. Let $f \in (\mathbb{C}^{MB})_\chi$ be bounded and s.t. $\underline{K}f = 0$ in $B(0, R)$ so that in $A(\frac{R}{2}, R)$,*

$$f(z) = R_B(\psi_1(z)) + \overline{R_B}(\psi_2(z)) + \varepsilon(z)$$

where $\psi_1(z) = z^s \sum_{n \geq 0} a_n z^n$, $\psi_2(z) = \bar{z}^{-s} \sum_{n \geq 0} b_n \bar{z}^n$. Then if $\eta > 0$, for $z = O(1)$,

$$f(z) = \frac{b_0}{\tau 2^{-s} \Gamma(s)} f_\chi(z) + \frac{a_0}{\tau 2^s \Gamma(-s)} h_\chi(z) + O(R^{s+\eta} \|\varepsilon\|_{\infty, A(\frac{R}{2}, R)} + R^{3s-1+\eta} \|\psi_i\|_{\infty, A(\frac{R}{2}, R)})$$

where $h_\chi(z) = \overline{f_{1, \bar{\chi}}(z)}$, and $\|\psi_i\| = \|\psi_1\| + \|\psi_2\|$.

Proof. Let $\phi \in (\mathbb{C}^{MB})_\chi$ be such that $\underline{K}\phi = 0$ in $B(0, r)$,

$$\phi(z) = \alpha f_\chi(z) + \beta h_\chi(z) + R_B(\phi_1(z)) + \overline{R_B}(\phi_2(z)) + \varepsilon(z)$$

in $A(\frac{r}{2}, r) = B(0, r) \setminus B(0, \frac{r}{2})$, say. Here ϕ_1 (resp. ϕ_2) is holomorphic (resp. antiholomorphic) and χ -multivalued, with Laurent expansions $\phi_1(z) = z^s \sum_{n \geq 1} a_n z^n$, $\phi_2(z) = \bar{z}^{-s} \sum_{n \geq 1} b_n \bar{z}^n$. Observe that

$$\phi_1(z) = \frac{1}{2i\pi} \oint_{C(0, \frac{3}{4}r)} \left(\frac{z}{w}\right)^s \frac{\phi_1(w)dw}{z-w} = \frac{1}{2i\pi} \oint_{C(0, \frac{3}{4}r)} \left(\frac{z}{w}\right)^s \frac{\phi_1(w)zdw}{w(z-w)}$$

for $z \in D(0, \frac{3}{4}r)$, and thus $\|\phi_1\|_{\infty, C(0, \varepsilon_0 r)} \leq c(\varepsilon_0) \|\phi_1\|_{\infty, C(0, r)}$, where $c(\varepsilon_0) \sim \varepsilon_0^{s+1}$ goes to zero as ε_0 goes to zero ($\varepsilon_0 > 0$ small, to be fixed later). We have a similar estimate for ϕ_2 : $\|\phi_2\|_{\infty, C(0, \varepsilon_0 r)} \leq c'(\varepsilon_0) \|\phi_2\|_{\infty, C(0, r)}$, $c'(\varepsilon_0) \sim \varepsilon_0^{s-1}$.

If γ_r is a simple cycle on Γ approximating $C(0, \frac{3}{4}r)$, let B be the set of black vertices on or inside γ_δ . We have the replication identity

$$\phi(z) = \sum_{w \in \gamma^W} \underline{K}(\phi \mathbf{1}_B)(w) \underline{K}_\chi^{-1}(z, w)$$

where γ^W are white vertices which are on γ_r or are outside of γ_r and adjacent to a black vertex on γ_r . For $z \in B(0, r/2)$, say, we have:

$$\phi(z) = \alpha f_\chi(z) + \beta h_\chi(z) + O(\|\varepsilon\|_{\infty, A(\frac{r}{2}, r)}(|z|/r)^{-s}) + \sum_{w \in \gamma^W} \mathbf{K}((R_B \phi_1 + \bar{R}_B \phi_2) \mathbf{1}_B)(w) \mathbf{K}_\chi^{-1}(z, w)$$

As in the surgery section, for $z \in A(\frac{\varepsilon_0 r}{2}, \varepsilon_0 r)$, we may estimate the discrete Cauchy integral by:

$$\begin{aligned} \sum_{w \in \gamma^W} \mathbf{K}((R_B \phi_1) \mathbf{1}_B)(w) \mathbf{K}_\chi^{-1}(z, w) &= R_B \frac{1}{2i\pi} \oint_{C(0, \frac{3}{4}r)} \left(\frac{z}{w}\right)^s \frac{\phi_1(w)dw}{z-w} + O(\|\phi_1\|_{\infty, A(\frac{r}{2}, r)} \varepsilon_0^{-s} r^{2s-1}) \\ \sum_{w \in \gamma^W} \mathbf{K}((\bar{R}_B \phi_2) \mathbf{1}_B)(w) \mathbf{K}_\chi^{-1}(z, w) &= \bar{R}_B \frac{-1}{2i\pi} \oint_{C(0, \frac{3}{4}r)} \left(\frac{\bar{z}}{\bar{w}}\right)^{-s} \frac{\phi_2(w)d\bar{w}}{\bar{z}-\bar{w}} + O(\|\phi_2\|_{\infty, A(\frac{r}{2}, r)} \varepsilon_0^{-s} r^{2s-1}) \end{aligned}$$

Consequently in $A(\frac{\varepsilon_0 r}{2}, \varepsilon_0 r)$, we have

$$\phi(z) = \alpha f_\chi(z) + \beta h_\chi(z) + R_B(\phi_1(z)) + \bar{R}_B(\phi_2(z)) + \varepsilon(z)$$

with

$$\|\phi_i\|_{\infty, A(\frac{\varepsilon_0 r}{2}, \varepsilon_0 r)} \leq c(\varepsilon_0) \|\phi_i\|_{\infty, A(\frac{r}{2}, r)}$$

for $i = 1, 2$, and

$$\|\varepsilon\|_{\infty, A(\frac{\varepsilon_0 r}{2}, \varepsilon_0 r)} \leq c\varepsilon_0^{-s} \left(\|\varepsilon\|_{\infty, A(\frac{r}{2}, r)} + \|\phi_i\|_{\infty, A(\frac{r}{2}, r)} r^{2s-1} \right)$$

Fix $\eta > 0$ arbitrarily small. For small enough ε_0 , $k \geq 1$, we get

$$\|\phi_1\|_{\infty, A(\frac{\varepsilon_0^k r}{2}, \varepsilon_0^k r)} \leq (\varepsilon_0^k)^{s+1-\eta} \|\phi_1\|_{\infty, A(\frac{r}{2}, r)}, \quad \|\phi_2\|_{\infty, A(\frac{\varepsilon_0^k r}{2}, \varepsilon_0^k r)} \leq (\varepsilon_0^k)^{-s+1-\eta} \|\phi_2\|_{\infty, A(\frac{r}{2}, r)}$$

and

$$\varepsilon_0^{s+\eta} \|\varepsilon\|_{\infty, A(\frac{\varepsilon_0^{k+1} r}{2}, \varepsilon_0^{k+1} r)} \leq \|\varepsilon\|_{\infty, A(\frac{\varepsilon_0^k r}{2}, \varepsilon_0^k r)} + \|\phi_i\|_{\infty, A(\frac{r}{2}, r)} r^{2s-1} (\varepsilon_0^k)^{2s-1+1-s-\eta}$$

so that for η small enough we get

$$\|\varepsilon\|_{\infty, A(\frac{\varepsilon_0^k r}{2}, \varepsilon_0^k r)} \leq (\varepsilon_0^k)^{-s-\eta} \left(\|\varepsilon\|_{\infty, A(\frac{r}{2}, r)} + C \|\phi_i\|_{\infty, A(\frac{r}{2}, r)} r^{2s-1} \right)$$

and for $z = O(1)$:

$$\phi(z) = \alpha f_\chi(z) + \beta h_\chi(z) + O(r^{s+\eta} \|\varepsilon\|_{\infty, A(\frac{r}{2}, r)} + r^{3s+\eta-1} \|\phi_i\|_{\infty, A(\frac{r}{2}, r)} + r^{s-1} \|\phi_i\|_{\infty, A(\frac{r}{2}, r)})$$

We have

$$\begin{aligned} f_\chi(z) &= \bar{R}_B(\bar{\tau} 2^{-s} \Gamma(s) \bar{z}^{-s}) + O(|z|^{s-1}) \\ h_\chi(z) &= R_B(\tau 2^s \Gamma(-s) z^s) + O(|z|^{-s-1}) \end{aligned}$$

Taking f as in the statement of the lemma, set $\phi_1(z) = \psi_1(z) - a_0 z^s$, $\phi_2(z) = \psi_2(z) - b_0 \bar{z}^{-s}$, $\alpha = \frac{b_0}{\bar{\tau} 2^{-s} \Gamma(s)}$, $\beta = \frac{a_0}{\tau 2^s \Gamma(-s)}$. We get

$$f(z) = \alpha f_\chi(z) + \beta h_\chi(z) + R_B(\phi_1(z)) + \bar{R}_B(\phi_2(z)) + \varepsilon'(z)$$

where $\|\phi_i\|_{\infty, A(\frac{R}{2}, R)}$, αR^{-s} and βR^s are of order $\|\psi_i\|_{\infty, A(\frac{R}{2}, R)}$ and

$$\|\varepsilon'\|_{\infty, A(\frac{R}{2}, R)} = O(\|\varepsilon\|_{\infty, A(\frac{R}{2}, R)} + \|\psi_i\|_{\infty, A(\frac{R}{2}, R)} R^{2s-1})$$

which concludes. \square

This applies in particular to $f = \mathbf{K}_\chi^{-1}(\cdot, w)$, $R = |w|/2$. For $b = O(1)$, this leads to an estimate of $\mathbf{K}_\chi^{-1}(b, w)$ (which is *a priori* of order $O(R^{s-1})$) within $O(R^{3s-2+\eta})$. By replication, for $s < \frac{1}{4}$, if ϕ is discrete holomorphic and χ -multivalued in a ball $B(0, R)$, we can estimate $\phi(b)$ for b adjacent to the singularity based on values taken by ϕ in the neighbourhood of a closed cycle around 0 in $A(\frac{R}{2}, R)$.

7.2 Pair correlations

Let us now consider the situation where two points $x \neq y$ (midpoints of edges of Λ) are marked. We consider functions with monodromy $\chi = e^{2i\pi s}$, $s \in (0, \frac{1}{2})$, around x and $\bar{\chi}$ around y . We are interested in estimating S_ρ , a kernel inverting K operating on these functions, in particular near the diagonal.

More specifically, we consider a family of rhombus tilings $(\Lambda_\delta)_\delta$ with respective edge length δ , $\delta \searrow 0$ along some sequence. Two points x_δ, y_δ (midpoints of edges of Λ) are marked; we assume $x_\delta \rightarrow x$, $y_\delta \rightarrow y$. Let $\rho : \pi_1(\mathbb{C} \setminus \{x, y\}) \rightarrow \mathbb{U}$ be the unitary character s.t. $\rho(\gamma_x) = \chi$, $\rho(\gamma_y) = \bar{\chi}$ where γ_x (resp. γ_y) is a simple loop winding around x (resp. y) counterclockwise. Let $(\mathbb{C}^{M_B})_\rho$ (resp. $(\mathbb{C}^{M_V})_\rho$) be the space of multiplicatively multi-valued functions on M_B transforming according to ρ .

First we consider the discrete Laplacian operating on M_V .

Lemma 17. *We assume $x \neq y$, ρ non trivial.*

1. *There is no nonzero bounded harmonic function in $(\mathbb{C}^{M_V})_\rho$.*
2. *For $w \in M_V$, there is a unique bounded function $G_\rho^\delta(\cdot, w) \in (\mathbb{C}^{M_V})_\rho$ s.t. $\Delta_\Gamma G_\rho^\delta(\cdot, w) = \delta_w$.*
3. *There a unique bounded continuous ρ -multivalued function $G_\rho(\cdot, w)$ on $\mathbb{C} \setminus \{x, y\}$ s.t. $\Delta G_\rho(\cdot, w) = \delta_w$ and $G_\rho(z, w) \rightarrow 0$ as $z \rightarrow x, y$.*
4. *As $\delta \searrow 0$, $G_\rho^\delta(\cdot, \cdot)$ converges uniformly to $G_\rho(\cdot, \cdot)$ uniformly on compact subsets of $\{(z, w) \in (\mathbb{C} \setminus \{x, y\})^2, z \neq w\}$.*

Proof. 1. Let h be a bounded harmonic function in $(\mathbb{C}^{M_V})_\rho$. Fix $z \in M_V$. Consider the random walk (X_n) started from z stopped at τ , the time of first return to z or the first exit of $B(0, n)$, whichever comes first. Let w_x (resp. w_y) be the winding of (X_0, \dots, X_τ) around x (resp. y). An excursion decomposition of the random walk between successive visits to z before time τ shows that:

$$h(z) = \frac{1 - \mathbb{P}(X_\tau = z)}{1 - \mathbb{P}(X_\tau = x)\mathbb{E}(\chi^{w_x - w_y} | X_\tau = z)} \mathbb{E}(h(X_\tau) | X_\tau \neq z)$$

As $n \rightarrow \infty$, $|\mathbb{E}(\chi^{w_x - w_y} | X_\tau = z)|$ stays bounded away from 1 while $\mathbb{P}(X_\tau = z) \rightarrow 1$. Consequently $h(z) = 0$.

2. Uniqueness follows from 1. For existence, we set $h(z) = \mathbb{E}_z(\rho(\gamma))$, where γ is the concatenation of a random walk started from z stopped when it first hits w and a fixed path from w to z (one may also reason on a branched cover of M_V , stopping the walk when it hits a lift of w). Then h is bounded and harmonic except at w . It is indeed not harmonic at w by 1. Consequently it is proportional to $G_\rho(\cdot, w)$.
3. By compactifying at infinity, a bounded harmonic function has a removable singularity at infinity. By mapping $\{x, y\}$ to $\{0, \infty\}$ by a homography and lifting to the universal cover of $\mathbb{C} \setminus \{0\}$ via \log , the problem is to find h harmonic except on $w' + 2i\pi\mathbb{Z}$ s.t. $h(w' + 2i\pi k) = \chi h(w')$. It is then easy to see that the solution to this problem exists and is unique (up to normalisation) and decays as $\Re(z) \rightarrow \pm\infty$ (corresponding to x, y in the original problem).
4. Fix $\varepsilon > 0$; we consider $\{(z, w) \in (B(0, \varepsilon^{-1}) \setminus (B(x, 3\varepsilon) \cup B(y, 3\varepsilon)))^2 : |z - w| \geq \varepsilon\}$. We need to show uniform convergence of G_ρ^δ on this set.
Let γ_e, γ_i be simple paths on Γ_δ at distance $O(\delta)$ of the circles $C(w, 2\varepsilon), C(w, \varepsilon)$ respectively. Let B be the connected component of w in $\Gamma_\delta \setminus \gamma_e$. Let τ_e (resp. τ_i) be the time of first exit of $\Gamma_\delta \setminus \gamma_e$

(resp. $\Gamma_\delta \setminus \gamma_i$) by a random walk on Γ . We define (twisted) Poisson operators $P_e : \mathbb{C}^{\gamma_e} \rightarrow \mathbb{C}^{\gamma_i}$ and $P_i : \mathbb{C}^{\gamma_i} \rightarrow \mathbb{C}^{\gamma_e}$ by $(P_e f)(z) = \mathbb{E}_z(f(X_{\tau_e}))$ and

$$(P_i f)(z) = \mathbb{E}_z(\chi^N f(X_{\tau_i})),$$

where N is the algebraic number of crossings of a branch cut between x and y (not intersecting γ_e) by the random walk before τ_i . (One may also reason on a branched cover of M_V , lifting f to a ρ -multivalued function). We omit the dependence on δ for lightness of notation.

As before, one can show that $(P_e P_i)$ is a strict contraction on $L^\infty(\gamma_i)$, uniformly in w , δ small enough ($\varepsilon > 0$ is fixed). Moreover, by starting a random walk on γ_i and stopping it on its first return to γ_i after its first visit to γ_e , we get the following identity in $L^\infty(\gamma_i)$:

$$G_\rho^\delta(\cdot, w)|_{\gamma_i} = G_B^\delta(\cdot, w)|_{\gamma_i} + (P_e P_i)G_\rho^\delta(\cdot, w)|_{\gamma_i}$$

Consequently,

$$G_\rho^\delta(\cdot, w)|_{\gamma_i} = \left(\sum_{k=0}^{\infty} (P_e P_i)^k \right) G_B^\delta(\cdot, w)|_{\gamma_i},$$

a summable series in $L^\infty(\gamma_i)$.

Clearly, the analogue decomposition holds for the continuous Green kernel G_ρ . We know that $G_B^\delta(\cdot, w)|_{\gamma_i}$ converges uniformly to $G_B(\cdot, w)|_{\gamma_i}$ as $\delta \searrow 0$ and in particular is uniformly bounded. Since $(P_e P_i)$ is uniformly strictly contracting, we only need to establish uniform convergence of each term in the expansion.

We note that $P_e^\delta h$ converges to $P_e h$ in $L^\infty(\gamma_i(w))$ uniformly in h, w as $\delta \searrow 0$, where h is (the restriction to $\gamma_e(w)$) of a 1-Lipschitz function. This may be shown by contradiction, using equicontinuity (of $h|_{\gamma_i}$ and of $P_e^\delta h$ on compact subsets of B).

Similarly, $P_i^\delta h$ converges to $P_i h$ in $L^\infty(\gamma_e(w))$ uniformly in h, w , where h is (the restriction to $\gamma_e(w)$) of a 1-Lipschitz function. We reason as above, the contradiction coming from the uniqueness in the following Dirichlet problem: find h_0 harmonic, bounded, ρ -multivalued in $\mathbb{C} \setminus (\{x, y\} \cup C(w, \varepsilon))$, with boundary condition a continuous function h on $C(0, \varepsilon)$. As in 3., this may be recast as a standard Dirichlet problem in \mathbb{C} .

Using iteratively these convergence statements for P_e^δ, P_i^δ , we obtain termwise convergence in the series expansion for G_ρ^δ above.

□

As was the case with one singularity, we can use G_ρ^δ to construct an inverting kernel \hat{S}_ρ for \mathbf{K} operating on $(\mathbb{C}^{M_B})_\rho$. Among such kernels, it is characterised by the fact that $\hat{S}_\rho(\cdot, w)$ restricted to M_V is harmonic except at endpoints of the edge of Γ corresponding to w . The restriction of \hat{S}_ρ to M_V is $G_\rho^\delta(\cdot, x') - G_\rho^\delta(\cdot, x)$, where $w \in M_W$ corresponds to the oriented edge (xx') of Γ_δ . On M_F , it is deduced by harmonic conjugation; it vanishes on the vertices of Γ^\dagger corresponding to x, y respectively. A discrete Green's formula argument (cutting the domain along a branch cut from x to y) shows that these two conditions for $\hat{S}_\rho(\cdot, w)|_{M_F}$ are consistent. Indeed, the variation of the harmonic conjugate between these two vertices is proportional to the flux of \hat{S}_ρ across a branch cut connecting the vertices; as \hat{S}_ρ decays at infinity, this flux is the integral of the Laplacian of \hat{S}_ρ in the fundamental domain defined by the cut, hence zero. Note also that \hat{S}_ρ is not invariant under duality $\Gamma \leftrightarrow \Gamma^\dagger$ (ie reasoning on the random walk on Γ^\dagger produces $\hat{S}_\rho^\dagger \neq \hat{S}_\rho$).

In order to estimate \hat{S}_ρ , let us evaluate $\partial_w G_\rho, \partial_{\bar{w}} G_\rho$ (in the continuous limit). As a function of z , $\partial_w G_\rho$ is locally the sum of a meromorphic and an antimeromorphic component; the decomposition is unique up to an additive constant. The additive constant can be specified uniquely in order to make these two components individually ρ -multivalued. Moreover, $G_\rho(z, w) - \frac{2}{\pi} \log |z - w|$ is harmonic in z in a neighbourhood of w . If

$\chi = e^{2i\pi s}$, $s \in (0, 1)$, this leaves the only possibility:

$$\partial_w G_\rho(z, w) = \left(\frac{z-x}{w-x} \right)^s \left(\frac{z-y}{w-y} \right)^{1-s} \frac{1}{\pi(z-w)}$$

as there is no (nonvanishing) ρ -multivalued, bounded holomorphic (resp. antiholomorphic) function on $\mathbb{C} \setminus \{x, y\}$.

We note that $G_\rho^\delta(z, w) = \overline{G_\rho^\delta(w, z)}$ and thus we also obtain convergence of discrete derivatives of $G_\rho^\delta(z, w)$ in either variable.

We are seeking an inverting kernel S_ρ which vanishes at infinity: $S_\rho(b, w) \rightarrow 0$ as $b \rightarrow \infty$. Such a kernel differs from \hat{S}_ρ by a discrete holomorphic function (freezing the second variable).

Observing that the restriction to M_V of a ρ -multivalued discrete holomorphic function is harmonic except possibly at the two vertices adjacent to x, y , we conclude that the space of bounded ρ -multivalued discrete holomorphic functions is at most two-dimensional. Let us construct such functions.

Let γ_δ be a simple cycle on Γ at distance $O(\delta)$ of $\gamma = C(x, |x-y|/2)$. Let γ_δ^\dagger be the outer boundary of the union of faces of Γ^\dagger corresponding to vertices of Γ on γ_δ . Set $f(z) = 2^s \Gamma(s)^{-1} f_{\chi, x}(z)$ for $z \in M_B$ inside or on γ_δ and $f(z) = 0$ otherwise. Then:

$$g_x(z) = f(z) - \sum_{w \in M_W} (\mathbf{K}f)(w) \hat{S}_\rho(z, w)$$

is ρ -multivalued and discrete holomorphic. Moreover $\mathbf{K}f$ is supported on vertices of M_W corresponding to edges of γ_δ and γ_δ^\dagger . Assume here $s \in (\frac{1}{2}, 1)$ (so that the dominant terms are holomorphic rather than antiholomorphic; otherwise conjugate). For $z \in M_V$, we have

$$\begin{aligned} \sum_{w \in \gamma_\delta} (\mathbf{K}f)(w) \hat{S}_\rho(z, w) &= \int_{\gamma_\delta} i(w-x)^{s-1} d_w G_\rho(z, w) + o(1) \\ \sum_{w \in \gamma_\delta^\dagger} (\mathbf{K}f)(w) \hat{S}_\rho(z, w) &= \int_{\gamma_\delta^\dagger} (w-x)^{s-1} * d_w G_\rho(z, w) + o(1) \end{aligned}$$

Taking into account $(w-x)^{s-1} = (\overline{w-x})^{1-s} r^{2s-2}$ on $C(x, r)$, we may write these integrals as contour integrals of closed forms and deform them to γ to obtain

$$\begin{aligned} g_x(z) &= (z-x)^{s-1} \chi_{|z-x| \leq |x-y|/2} \\ &\quad - \oint_\gamma (w-x)^{s-1} \left(\left(\frac{z-x}{w-x} \right)^s \left(\frac{z-y}{w-y} \right)^{1-s} \frac{dw}{\pi(z-w)} \right) + o(1) \\ &= (z-x)^{s-1} \chi_{|z-x| \leq |x-y|/2} - (- (z-x)^{s-1} \left(\frac{z-y}{x-y} \right)^{1-s} + (z-x)^{s-1} \chi_{|z-x| \leq |x-y|/2}) + o(1) \\ &= (z-x)^{s-1} \left(\frac{z-y}{x-y} \right)^{1-s} + o(1) \end{aligned}$$

by the residue formula. Symmetrically, one may construct another discrete holomorphic function with:

$$g_y(z) = (\overline{z-y})^{s-1} \overline{\left(\frac{z-x}{y-x} \right)^{1-s}} + o(1)$$

The asymptotics on M_F are obtained by harmonic conjugation (or by a similar argument), the additive constant being fixed by the condition on ρ -multivaluedness. Convergence is uniform on compact subsets

away from x, y, γ . By writing the Cauchy formula on another circle γ' , one obtains uniform convergence on compact subsets of $\mathbb{C} \setminus \{x, y\}$.

Thus one may add a linear combination of g_x, g_y to $\hat{S}_\rho(\cdot, w)$ to obtain a kernel vanishing at infinity. Indeed, a function on M_V (resp. M_F) which is bounded and (discrete) harmonic in a neighbourhood of infinity has a limit at infinity, as is easily shown eg by a random walk coupling argument. Convergence on compact sets is enough to ensure that

$$\lim_{z \rightarrow \infty, z \in M_V} g_x(z) = (x - y)^{s-1} + o(1)$$

Similarly, $\lim_{z \rightarrow \infty, z \in M_F} g_x(z) = i(x - y)^{s-1} + o(1)$, $\lim_{z \rightarrow \infty, z \in M_V} g_y(z) = (\overline{y - x})^{s-1} + o(1)$, $\lim_{z \rightarrow \infty, z \in M_F} g_y(z) = -i(\overline{y - x})^{s-1} + o(1)$. Thus, for δ small enough, one may find $a(w), b(w)$ s.t.

$$S_\rho(z, w) = \hat{S}_\rho(z, w) - a(w)g_x(z) - b(w)g_y(z)$$

vanishes at infinity. Combining with previous estimates on $\hat{S}_\rho(\cdot, w)$, g_x, g_y , we identify $a(w), b(w)$ up to $o(1)$.

Let us summarise the previous discussion.

Lemma 18. *For $s \in (-\frac{1}{2}, \frac{1}{2})$, δ small enough, for $w \in M_W$ there is a unique ρ -multivalued function $S_\rho(\cdot, w)$ vanishing at infinity s.t. $\mathbf{K}S_\rho(\cdot, w) = \delta_w$. Moreover,*

$$S_\rho(z, w) = \frac{1}{2}R_B \left(\left(\frac{z - x}{w - x} \cdot \frac{w - y}{z - y} \right)^s \frac{e^{i\nu(w)}}{\pi(z - w)} \right) + \frac{1}{2}\overline{R_B} \left(\overline{\left(\frac{z - x}{w - x} \cdot \frac{w - y}{z - y} \right)^s} \frac{e^{-i\nu(w)}}{\pi(\overline{z - w})} \right) + o(|x - y|^{-1})$$

with uniform convergence on compact subsets of $\{(z, w) \in (\mathbb{C} \setminus \{x, y\})^2 : z \neq w\}$.

The continuous holomorphic and antiholomorphic kernels are specified by the fact that they vanish as $z \rightarrow \infty$, and have singularities of order at most $|s|$ at x, y .

Let us consider a more general situation. Let $x_1, \dots, x_n, y_1, \dots, y_n$ be marked points, $s_1, \dots, s_n \in (0, 1)$. We consider $\rho : \pi_1(\mathbb{C} \setminus \{x_1, \dots, y_n\}) \rightarrow \mathbb{U}$ the unitary character such that $\rho(\gamma_{x_j}) = \rho(\gamma_{y_j})^{-1} = \chi_j = e^{2i\pi s_j}$ where γ_z is a counterclockwise loop around $z \in \{x_1, \dots, y_n\}$ with no other marked points in its interior. We are interested in the associated operator $(\mathbb{C}^{M_B})_\rho \rightarrow (\mathbb{C}^{M_W})_\rho$.

Let us address the case $n = 2$ using the surgery formalism (Lemma 8). Let γ be a simple loop with $\{x_1, y_1\}$ in its interior U_i and $\{x_2, y_2\}$ in its exterior U_o . Let ρ_1 (resp. ρ_2) be the character corresponding to weights $(s_1, 0)$ (resp. $(0, s_2)$). We have

$$S_{\rho_j}(z, w) = \frac{1}{2}R_B \left(\left(\frac{z - x_j}{w - x_j} \cdot \frac{w - y_j}{z - y_j} \right)^{s_j} \frac{e^{i\nu(w)}}{\pi(z - w)} \right) + \frac{1}{2}\overline{R_B} \left(\overline{\left(\frac{z - x_j}{w - x_j} \cdot \frac{w - y_j}{z - y_j} \right)^{s_j}} \frac{e^{-i\nu(w)}}{\pi(\overline{z - w})} \right) + o(|x - y|^{-1})$$

where all pairwise distances are of order 1. The Cauchy data spaces corresponding to the limiting continuous kernels are easy to identify:

$$\begin{aligned} C_i &= \{f|_\gamma : f(z) = g(z) \left(\frac{z - x_1}{w - x_1} \cdot \frac{w - y_1}{z - y_1} \right)^{s_1}, (g_{\bar{z}})|_{U_i} = 0\} \cap Lip(\gamma) \\ \bar{C}_i &= \{f|_\gamma : f(z) = g(z) \overline{\left(\frac{z - x_1}{w - x_1} \cdot \frac{w - y_1}{z - y_1} \right)^{s_1}}, (g_z)|_{U_i} = 0\} \cap Lip(\gamma) \\ C_o &= \{f|_\gamma : f(z) = g(z) \left(\frac{z - x_2}{w - x_2} \cdot \frac{w - y_2}{z - y_2} \right)^{s_2}, (g_{\bar{z}})|_{U_o} = 0, g(\infty) = 0\} \cap Lip(\gamma) \\ \bar{C}_o &= \{f|_\gamma : f(z) = g(z) \overline{\left(\frac{z - x_2}{w - x_2} \cdot \frac{w - y_2}{z - y_2} \right)^{s_2}}, (g_z)|_{U_o} = 0, g(\infty) = 0\} \cap Lip(\gamma) \end{aligned}$$

Note that in general $\bar{C}_i \neq \overline{C_i}$. We check that $C_i \cap C_0 = \bar{C}_i \cap C_0 = \{0\}$ and find the glued Cauchy kernels:

$$S_g(z, w) = \frac{1}{\pi(z-w)} \prod_{j=1}^2 \left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j}$$

$$\bar{S}_g(z, w) = \frac{1}{\pi(z-w)} \prod_{j=1}^2 \overline{\left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j}}$$

By induction on n (using surgery at each step to add a pair of insertions), we obtain

$$S_\rho(z, w) = \frac{1}{2} R_B \left(\frac{e^{i\nu(w)}}{\pi(z-w)} \prod_{j=1}^n \left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j} \right) + \frac{1}{2} \bar{R}_B \left(\frac{e^{-i\nu(w)}}{\pi(z-w)} \prod_{j=1}^n \overline{\left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j}} \right) + o(|x-y|^{-1})$$

with uniform convergence on compact subsets of $\Sigma^2 \setminus \Delta_\Sigma$, $\Sigma = \mathbb{C} \setminus \{x_1, \dots, y_n\}$. Through surgery, we also retain invertibility (ie S_ρ is uniquely characterised by $\mathbf{K} S_\rho(\cdot, w) = \delta_w$, $S_\rho(\cdot, w)$ vanishes at infinity), at least for small enough δ .

We would like to gain an explicit error estimate for S_ρ . The following is a simple (and rather crude) estimate, which will be enough for our purposes.

Lemma 19. *We have*

$$S_\rho(z, w) = \frac{1}{2} R_B \left(\frac{e^{i\nu(w)}}{\pi(z-w)} \prod_{j=1}^n \left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j} \right) + \frac{1}{2} \bar{R}_B \left(\frac{e^{-i\nu(w)}}{\pi(z-w)} \prod_{j=1}^n \overline{\left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j}} \right) + O(\delta^{1-2 \max_j |s_j|})$$

uniformly on compact sets of $\Sigma^2 \setminus \Delta_\Sigma$, $\Sigma = \mathbb{C} \setminus \{x_1, \dots, y_n\}$.

Proof. First we estimate $S_\rho(b, w)$ for w near a singularity, b in a compact set of Σ . From

$$\mathbf{K}_\chi^{-1}(b, w) = O(|w|^{-s})$$

for $\chi = e^{2i\pi s}$, $s \in (0, \frac{1}{2})$ (with singularity at 0), and the replication identity:

$$S_\rho(\cdot, w) = \mathbf{K}_{\chi_1, x_1}^{-1}(\cdot, w) \mathbf{1}_{B(x_1, r)} + S_\rho(\mathbf{K}(\mathbf{K}_{\chi_1, x_1}^{-1}(\cdot, w) \mathbf{1}_{B(x_1, r)} - \delta_w))$$

we get the estimate: $S_\rho(b, w) = O(|w|^{s_1-1})$ for w close to x_1 , b in a compact subset of Σ .

Now fix w_0 away from the singularities; we wish to estimate $S_\rho(\cdot, w_0)$. Let us set $\tilde{S}_\rho(\cdot, w) = \mathbf{K}^{-1}(\cdot, w_0) + \frac{1}{2} R_B(\alpha) + \frac{1}{2} \bar{R}_B(\beta)$ in $B(w_0, r)$; and

$$\tilde{S}_\rho(z, w_0) = \frac{1}{2} R_B \left(\frac{e^{i\nu(w)}}{\pi(z-w_0)} \prod_{j=1}^n \left(\frac{z-x_j}{w_0-x_j} \cdot \frac{w_0-y_j}{z-y_j} \right)^{s_j} \right) + \frac{1}{2} \bar{R}_B \left(\frac{e^{-i\nu(w)}}{\pi(z-w_0)} \prod_{j=1}^n \overline{\left(\frac{z-x_j}{w_0-x_j} \cdot \frac{w_0-y_j}{z-y_j} \right)^{s_j}} \right)$$

elsewhere (α, β, r to be fixed). We have

$$S_\rho(z, w) - \tilde{S}_\rho(z, w) = \sum_{w' \neq w_0} \mathbf{K}(\tilde{S}_\rho(\cdot, w_0)) S_\rho(\cdot, w_0)$$

On $\partial B(0, r)$, we have $\mathbf{K}(\tilde{S}_\rho(\cdot, w_0)) = O(\delta r^2)$ for an appropriate choice of α, β . For $r < |w-w_0| < 1$, we have $\mathbf{K}(\tilde{S}_\rho(\cdot, w_0))(w) = O(\delta^4 |w-w_0|^{-4})$. For w in a compact subset of $\Sigma \setminus \{w_0\}$, we have $\mathbf{K}(\tilde{S}_\rho(\cdot, w_0))(w) = O(\delta^4)$. For $|w-w_0| \gg 1$, $\mathbf{K}(\tilde{S}_\rho(\cdot, w_0))(w) = O(\delta^4 |w-w_0|^{-4})$ (ensuring convergence of the RHS in the replication

identity). Finally, for w close to x_j , $\mathbf{K}((\tilde{S}_\rho(\cdot, w_0))(w) = O(\delta^4|w - x_j|^{-|s_j|-3})$ (and similarly near y_j). This leads to:

$$\begin{aligned} S_\rho(z, w) - \tilde{S}_\rho(z, w) &= O((r\delta^{-1})(\delta r^2)r^{-1}) + O(\delta^{-2}\delta^4) + O\left(\sum_{k=\delta^{-1}}^{\infty} k(\delta^4(k\delta)^{-4})(k\delta)^{-1}\right) \\ &\quad + \sum_{j=1}^n O\left(\sum_{k=1}^{\delta^{-1}} k\delta^4(k\delta)^{-|s_j|-3}(k\delta)^{-|s_j|}\right) \end{aligned}$$

Thus setting $r = \sqrt{\delta}$, we get

$$S_\rho(z, w) - \tilde{S}_\rho(z, w) = O(\delta^{1-2\max_j |s_j|})$$

as stated. \square

We will need to analyse $S_\rho(\cdot, w)$ when w is adjacent to a singularity. This is obtained by a combination of Lemma 16 and the estimates we obtained for S_ρ in $\Sigma^2 \setminus \Delta_\Sigma$. Let us consider the ‘‘Robin kernels’’:

$$\begin{aligned} r_\rho(w) &= \lim_{z \rightarrow w} \left(\frac{1}{\pi(z-w)} \prod_{j=1}^n \left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j} - \frac{1}{\pi(z-w)} \right) = \frac{1}{\pi} \sum_{j=1}^n \left(\frac{s_j}{w-x_j} - \frac{s_j}{w-y_j} \right) \\ \bar{r}_\rho(w) &= \lim_{z \rightarrow w} \left(\frac{1}{\pi(z-w)} \prod_{j=1}^n \overline{\left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{-s_j}} - \frac{1}{\pi(z-w)} \right) = \frac{1}{\pi} \sum_{j=1}^n \left(\frac{-s_j}{w-x_j} - \frac{-s_j}{w-y_j} \right) \end{aligned}$$

(so that $\bar{r}_\rho(w) = \overline{r_\rho(w)}$).

Lemma 20. *If b, w are adjacent to the singularity $x \in \{x_1, \dots, y_n\}$, $s = \pm s_k$ the corresponding exponent ($s = s_k$ for $x = x_k$ and $s = -s_k$ for $x = y_k$), and $b \in \Gamma$, we have*

$$S_\rho(b, w) - \mathbf{K}_{\chi, x}^{-1}(b, w) = \frac{2s}{1-\chi} \left(\frac{x-b}{w-x} \right)^s \Re \left(i e^{i\nu(w)} z \right) + O(|x-y|^{-1-\varepsilon})$$

where $\varepsilon = \varepsilon(s)$ is positive for s small enough and

$$z = -\frac{s_k}{x_k - y_k} + \sum_{j \neq k} \left(\frac{s_j}{x_k - x_j} - \frac{s_j}{x_k - y_j} \right) = \lim_{w \rightarrow x} \left(\pi r_\rho(w) - \frac{s}{w-x} \right)$$

Proof. Let w be adjacent to the singularity $x = x_k$ (which is in a face of M); by translating we may assume $x_k = 0$. Let r be of order 1 and small enough so that other singularities are outside of $\bar{B}(0, r)$. Also set $s = s_k$, $\chi = e^{2i\pi s}$. We consider $f(b) = S_\rho(b, w) - \mathbf{K}_\chi^{-1}(b, w) \mathbf{1}_{B(0, r)}$ which is discrete holomorphic except near $\partial B(0, r)$ and is ρ -multivalued. The goal is to estimate $f(b)$ for $b = O(\delta)$ (in particular for b adjacent to the singularity). We have the Cauchy integral formula

$$f(b) = \sum_{w'} S_\rho(w', b) \mathbf{K}(f)(w') = \sum_{w' \neq w} S_\rho(w', b) \mathbf{K}(\mathbf{K}_\chi^{-1}(\cdot, w) \mathbf{1}_{B(0, r)})(w')$$

for $b \in B(0, r)$. From Lemma 15 and estimates on S_ρ , we obtain for $b \in A(\frac{r}{4}, \frac{r}{2})$:

$$\begin{aligned} f(z) &= \frac{\Gamma(1-s)e^{i\nu(w)}}{2} R_B \frac{1}{\pi(z-w)} \left(\prod_{j=1}^n \left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j} - \left(\frac{z-x_k}{w-x_k} \right)^{s_k} \right) \\ &\quad + \frac{\Gamma(1+s)e^{-i\nu(w)}}{2} \overline{R_B} \frac{1}{\overline{\pi(z-w)}} \left(\prod_{j=1}^n \overline{\left(\frac{z-x_j}{w-x_j} \cdot \frac{w-y_j}{z-y_j} \right)^{s_j}} - \overline{\left(\frac{z-x_k}{w-x_k} \right)^{s_k}} \right) \\ &\quad + O(\delta^{2-|s|} + O(\delta^{1-|s|-2\max_j |s_j|})) \\ &= R_B(\psi_1(z)) + \overline{R_B}(\psi_2(z)) + O(\delta^{1-3\max_j |s_j|}) \end{aligned}$$

where $\psi_1(z) = z^s \sum_{n \geq 0} a_n z^n$, $\psi_2(z) = \bar{z}^{-s} \sum_{n \geq 0} b_n \bar{z}^n$,

$$\begin{aligned} a_0 &= \frac{\Gamma(1-s)e^{i\nu(w)}}{2\pi w^s} \left(-\frac{s_k}{x_k - y_k} + \sum_{j \neq k} \left(\frac{s_j}{x_k - x_j} - \frac{s_j}{x_k - y_j} \right) \right) (1 + O(\delta)) \\ b_0 &= \frac{\Gamma(1+s)e^{-i\nu(w)}}{2\pi \bar{w}^{-s}} \left(\frac{s_k}{x_k - y_k} + \sum_{j \neq k} \left(\frac{-s_j}{x_k - x_j} - \frac{-s_j}{x_k - y_j} \right) \right) (1 + O(\delta)) \end{aligned}$$

and by Lemma 16, we conclude:

$$f(b) = \frac{b_0}{\bar{\tau} 2^{-s} \Gamma(s)} f_\chi(b) + \frac{a_0}{\tau 2^s \Gamma(-s)} h_\chi(z) + O(\delta^{1-4\max_j |s_j| - \eta})$$

for $b = O(1)$, $\eta > 0$ fixed. Taking into account (Lemma 10), we have for b adjacent to the singularity:

$$\begin{aligned} f_\chi(b) &= \frac{2i\pi e^{i\nu(b)}}{\bar{\chi} - 1} \overline{2(x-b)^{1-s}} \\ h_\chi(b) &= \overline{f_{1,\bar{\chi}}(b)} = -\frac{2i\pi e^{-i\nu(b)}}{\bar{\chi} - 1} (2(x-b))^{1+s} \end{aligned}$$

and $\tau = \frac{2}{\delta}(b-x)$ if $b \in \Gamma$ adjacent to x . □

7.3 Variational analysis

The previous local estimates may be used to estimate

$$\langle \chi^{h(y)-h(x)} \rangle$$

for $x, y \in M^\dagger$ at large distance, where h is the height function associated with a matching \mathbf{m} of M and $\chi = e^{2i\pi s}$ is fixed ($s \in (0, \frac{1}{2})$). More precisely, we are interested in estimating $\langle \chi^{h(y')-h(x)} \rangle / \chi^{h(y)-h(x)}$ within $o(|y' - y|/|x - y|)$.

Fix $x \in M^\dagger$ and a simple path γ' from x to y' on M^\dagger ; the penultimate vertex of γ' is y : $\gamma' = (x, \dots, y, y')$. We also denote $\gamma = (x, \dots, y)$, the path stopped at y . Let $E_r = E_r(\gamma)$ be the set of edges of M crossed by γ , with the black vertex on the right hand side of γ ; and $E_l = E_l(\gamma)$ the edges crossed by γ , with the black vertex on the left hand side. Then ([33])

$$h(y) - h(x) = \frac{1}{2\pi} \text{wind}(\gamma) + \sum_{e \in E_r} \mathbf{1}_{e \in \mathbf{m}} - \sum_{e \in E_l} \mathbf{1}_{e \in \mathbf{m}}$$

In order to define the winding number of γ , we draw it as a succession of smooth arcs connecting midpoints of segments (bb') (ie centers of faces of M), where $b \in \Gamma$ and $b' \in \Gamma^\dagger$, in such a way that γ crosses these segments normally.

Let $K_\gamma : M_B \rightarrow M_W$ be the operator defined by $K_\gamma(b, w) = \bar{\chi}K(b, w)$ if $(bw) \in E_r(\gamma)$; $K_\gamma(b, w) = \chi K(b, w)$ if $(bw) \in E_l(\gamma)$; and $K_\gamma(b, w) = K(b, w)$ otherwise.

Let us remark that if $\tilde{\gamma}$ is another path from x to y , then $K_{\tilde{\gamma}}$ is conjugate to K_γ by diagonal matrices involving the winding number of the loop obtained by concatenating γ with $\tilde{\gamma}$ (in reverse orientation). Moreover, by using γ as a branch cut (lifting functions on M to functions in $(\mathbb{C}^M)_\rho$, where ρ is the unitary character of $\pi_1(\mathbb{C} \setminus \{x, y\})$ defined as before), one can identify K_γ with K operating on $(\mathbb{C}^{M_B})_\rho$. Correspondingly, we get an inverting kernel $S_\gamma : \mathbb{C}^{M_W} \rightarrow \mathbb{C}^{M_B}$.

Lemma 21. $K_\gamma \underline{K}^{-1} : M_W \rightarrow M_W$ is a finite rank perturbation of the identity and

$$\langle \chi^{h(y)-h(x)} \rangle = e^{is \cdot \text{wind}(\gamma)} \det(K_\gamma \underline{K}^{-1})$$

Proof. See the proof of Lemma 2. □

We now analyse the effect of taking the last step along γ' . Let (bw) be the edge of M separating $y, y' \in M^\dagger$. Recall that $S_\gamma(., w)$ can be characterized as the only function in \mathbb{C}^{M_B} vanishing at infinity such that $K_\gamma S_\gamma(., w) = \delta_w$; $S_{\gamma'}(., w)$ is characterised similarly. Since $K_{\gamma'} - K_\gamma$ is supported on w and

$$(K_\gamma S_{\gamma'}(., w))(w) = 1 + (1 - \chi^{\pm 1})K(w, b)S_{\gamma'}(b, w)$$

(here $\chi^{\pm 1} = \chi$ (resp. χ^{-1}) if b is to the left (resp. to the right) of γ'), we deduce

$$S_{\gamma'}(., w) = (1 + (1 - \chi^{\pm 1})K(w, b)S_{\gamma'}(b, w))S_\gamma(., w)$$

In order to evaluate $\langle \chi^{h(y')-h(x)} \rangle / \langle \chi^{h(y)-h(x)} \rangle$, we have to evaluate $\det(K_{\gamma'} \underline{K}^{-1}) / \det(K_\gamma \underline{K}^{-1})$. Let us observe that $K_{\gamma'} S_\gamma = \text{Id} + (K_{\gamma'} - K_\gamma) S_\gamma$ is a rank 1 perturbation of the identity and that

$$(K_{\gamma'} S_\gamma)(K_\gamma \underline{K}^{-1}) = K_{\gamma'} \underline{K}^{-1}$$

as bounded operators in $L^2(M_W, \mu)$ where $\mu\{w\} = (1 + |w|)^{-\varepsilon}$ for some $\varepsilon > 0$. This follows from $S_\gamma K_\gamma = \text{Id}$ (indeed, $(S_\gamma K_\gamma)\delta_b - \delta_b$ is in the kernel of K_γ and vanishes at infinity, hence is identically zero). Consequently,

$$\frac{\det(K_{\gamma'} \underline{K}^{-1})}{\det(K_\gamma \underline{K}^{-1})} = \det(K_{\gamma'} S_\gamma) = \det(\text{Id} + (K_{\gamma'} - K_\gamma) S_\gamma) = 1 + (K_{\gamma'} - K_\gamma)(w, b)S_\gamma(b, w) = 1 + (\chi^{\pm 1} - 1)K(w, b)S_\gamma(b, w)$$

since $K_{\gamma'} - K_\gamma$ has rank 1. Then we use the key estimate (Lemma 20)

$$S_\rho(b, w) - \underline{K}_{\bar{\chi}, y}^{-1}(b, w) = \frac{2s^2}{1 - \chi} \left(\frac{w - y}{y - b} \right)^s \Re \left(i e^{i\nu(w)} / (y - x) \right) + O(|x - y|^{-1-\varepsilon})$$

together with $\pm iK(w, b) = y' - y$, $\frac{w-y}{y-b} = \frac{y-w}{y'-w}$ and the exact result

$$K(w, b) \underline{K}_{\bar{\chi}, y}^{-1}(b, w) = \frac{1}{1 - \chi} \left(1 - \left(\frac{y' - w}{y - w} \right)^{-s} \right)$$

to obtain

$$\frac{\det(K_{\gamma'} \underline{K}^{-1})}{\det(K_\gamma \underline{K}^{-1})} = \left(\frac{y' - w}{y - w} \right)^{-s} \left(1 - 2s^2 \Re \left(\frac{y' - y}{y - x} \right) \right) + O(|x - y|^{-1-\varepsilon})$$

Thus we can state

Proposition 22. *For s small enough, there exists a $c = c(\Lambda, s) > 0$ such that*

$$|\langle \chi^{h(y)-h(x)} \rangle| \sim c|x-y|^{-2s^2}$$

Proof. From the previous computation we get

$$\left| \frac{\langle \chi^{h(y')-h(x)} \rangle \cdot |y-x|^{-2s^2}}{\langle \chi^{h(y)-h(x)} \rangle \cdot |y'-x|^{-2s^2}} \right| = 1 + O(|x-y|^{-1-\varepsilon})$$

with $\varepsilon > 0$ for s small enough. Thus if we fix x and let y go to infinity, we get

$$|\langle \chi^{h(y)-h(x)} \rangle| \sim c(x, \Lambda, s)|x-y|^{-2s^2}$$

Since $|\langle \chi^{h(y)-h(x')} \rangle| / |\langle \chi^{h(y)-h(x)} \rangle| = 1 + O(|x-x'|/|x-y|)$, we conclude that c does not depend on x . \square

More generally, if $\{z_1, \dots, z_{2n}\} = \{x_1, \dots, y_n\}$ are marked points, $s_1, \dots, s_{2n} > 0$ are small enough exponents with $s_1 + s_2 = \dots = s_{2n-1} + s_{2n} = 0$, we have (by “integrating” Lemma 20)

$$\langle \exp(2i\pi \sum_{j=1}^{2n} s_j h(z_j)) \rangle \sim c \prod_{i < j} |z_i - z_j|^{2s_i s_j}$$

in agreement with the electric vertex correlator heuristic. The pairing of insertions $(z_1, z_2), \dots, (z_{2n-1}, z_{2n})$ is somewhat restrictive. For instance, it is not obvious how to treat correlators such as

$$\langle \exp(\frac{2i\pi}{3}(h(x) + h(y) + h(z))) \rangle$$

or even what to expect (as compactification may start to play a role). Indeed, up to a phase, one can write

$$\langle \exp(\frac{2i\pi}{3}(h(x) + h(y) + h(z))) \rangle = \langle \exp(\frac{2i\pi}{3}(h(x) - h(z) + h(y) - h(z))) \rangle$$

and if z' is close to, but at macroscopic distance of, z , we have asymptotically

$$\langle \exp(\frac{2i\pi}{3}(h(x) - h(z) + h(y) - h(z'))) \rangle \sim |(x-z)(x-z')(y-z)(y-z')|^{-\frac{2}{9}} |(x-y)(z-z')|^{\frac{2}{9}}$$

Taking $z' \rightarrow z$ yields the heuristic

$$\langle \exp(\frac{2i\pi}{3}(h(x) + h(y) + h(z))) \rangle \stackrel{??}{\sim} |(x-z)(y-z)|^{-\frac{4}{9}} |x-y|^{\frac{2}{9}}$$

which is plainly incorrect (as it is not symmetric in x, y, z).

We turn to the case where s is not small, and for simplicity we first discuss the case of two marked points. Remark that all previous estimates are uniform in s for s in a compact interval of $(0, \frac{1}{2})$. Again we use a variational argument; now x, y are fixed and s is varying. Starting from $|\langle \chi^{h(y)-h(x)} \rangle| = |\det(K_\gamma \underline{K}^{-1})|$, where $\chi = e^{2i\pi s}$ and K_γ is implicitly a function of s , we get

$$\frac{d}{ds} \log |\langle \chi^{h(y)-h(x)} \rangle| = \Re \text{Tr}(\dot{K}_\gamma S_\gamma)$$

and $\dot{K}_\gamma(w, b) = \pm 2i\pi \chi^{\pm 1} \mathbf{K}(w, b)$ if γ crosses the edge (wb) . Hence we need to evaluate $S_\gamma(w, b)$ for these edges. We have (uniformly in (z, w) in a compact subset of $\{(z, w) : z \neq x, y; w \neq x, y; z \neq w\}$)

$$S_\gamma(z, w) = \frac{1}{2} R_B \left(\left(\frac{z-x}{z-y} \right)^s \left(\frac{w-x}{w-y} \right)^{-s} \frac{e^{i\nu(w)}}{\pi(z-w)} \right) + \frac{1}{2} \overline{R_B} \left(\overline{\left(\frac{z-x}{z-y} \right)^{-s} \left(\frac{w-x}{w-y} \right)^s} \frac{e^{-i\nu(w)}}{\pi(\overline{z-w})} \right) + o(|x-y|^{-1})$$

where $\left(\frac{u-x}{u-y}\right)^{s'} = \exp(s' \log((u-x)/(u-y)))$ and $u \mapsto \log((u-x)/(u-y))$ is a determination branched along γ . Since

$$\left(\frac{z-x}{z-y}\right)^s \left(\frac{w-x}{w-y}\right)^{-s} \frac{1}{z-w} = \frac{1}{z-w} + s \left(\frac{1}{w-x} - \frac{1}{w-y}\right) + O(|z-w|/|w-x|^2 + |z-w|/|w-y|^2)$$

we obtain if $b \sim w$

$$S_\gamma(b, w) - \underline{\mathbf{K}}^{-1}(b, w) = \frac{s}{2\pi} R_B \left(e^{i\nu(w)} \left(\frac{1}{w-x} - \frac{1}{w-y} \right) \right) - \frac{s}{2\pi} \overline{R_B} \left(e^{-i\nu(w)} \left(\frac{1}{\overline{w-x}} - \frac{1}{\overline{w-y}} \right) \right) + o(1/|x-y|)$$

except if (bw) crosses γ , in which case the LHS is replaced with $\chi^{\pm 1}(S_\gamma(b, w) - \underline{\mathbf{K}}^{-1}(b, w))$.

Let γ' be a subpath of γ running from x' to y' , and $E(\gamma')$ the set of edges of M crossed by γ' . If $(vv') = (wb)^\dagger$ (as oriented edges), we have $K(w, b) = i(v - v')$ and $\mathbf{K}(w, b) = e^{-i\nu(w) - i\nu(b)} i(v - v')$. Thus if v_i, v_{i+1} are consecutive points on γ' corresponding to the unoriented edge (wb) ,

$$\pm \mathbf{K}(w, b)(R_B(e^{i\nu(w)}\alpha) + \overline{R_B}(e^{-i\nu(w)}\beta)) = i\alpha(v_i - v_{i+1}) - i\beta(\overline{v_i - v_{i+1}}) = -i\alpha \int_{v_i}^{v_{i+1}} dz + i\beta \int_{v_i}^{v_{i+1}} d\bar{z}$$

where $\pm = +$ if $(bw) \in E_r(\gamma)$ and $-$ if $(bw) \in E_l(\gamma)$. Then

$$\begin{aligned} \Re \sum_{(bw) \in E(\gamma')} \dot{K}_\gamma(b, w) S_\gamma(b, w) &= -s \Re \int_{x'}^{y'} \left(\frac{1}{z-x} - \frac{1}{z-y} \right) dz - s \Re \int_{x'}^{y'} \left(\frac{1}{\overline{z-x}} - \frac{1}{\overline{z-y}} \right) d\bar{z} + o(1) \\ &= 2s \Re \log \left(\frac{(y' - x)(x' - y)}{(x' - x)(y' - y)} \right) + o(1) \end{aligned}$$

(one may also keep track of the imaginary part, since $\mathbf{K}(w, b)\underline{\mathbf{K}}^{-1}(b, w) = p(w, b)$ is expressed in terms of the local geometry). We still need to address the logarithmic singularities at the endpoints of γ . A discrete Cauchy formula together with estimates on $S_\rho(z, w)$ for z close to x and $|w-x|$ comparable to $|x-y|$ leads to the estimate

$$S_\rho(z, w) - \underline{\mathbf{K}}_{\chi, x}^{-1}(z, w) = O\left(\frac{|x-y|^{s-1}}{|w-x|^s} \cdot \frac{|x-y|^s}{|w-x|^s}\right) = O(|x-y|^{2s-1}|w-x|^{-2s})$$

for $|w-x| \leq \frac{1}{4}|x-y|$, $\frac{1}{2}|w-x| \leq |z-x| \leq 2|w-x|$. The same estimate holds when interverting x and y . Consequently, if we choose γ such that the lengths of its segments are comparable to their Euclidean length, and γ_i is the initial segment of γ (between x and x'), we have

$$\sum_{(bw) \in E(\gamma_i)} \dot{K}_\gamma(b, w) S_\gamma(b, w) = \sum_{(bw) \in E(\gamma_i)} \dot{K}_\gamma(b, w) \underline{\mathbf{K}}_{\chi, x}^{-1}(b, w) + O((|x' - x|/|x - y|)^{1-2s})$$

From the estimate

$$\underline{\mathbf{K}}_{\chi, x}^{-1}(z, w) = \frac{1}{2} R_B \left(\left(\frac{z-x}{w-x} \right)^s \frac{e^{i\nu(w)}}{\pi(z-w)} \right) + \frac{1}{2} \overline{R_B} \left(\left(\frac{z-x}{w-x} \right)^{-s} \frac{e^{-i\nu(w)}}{\pi(\overline{z-w})} \right) + O(|w-x|^{2s-2})$$

for $\frac{1}{2}|w-x| \leq |z-x| \leq 2|w-x|$, we obtain

$$\underline{\mathbf{K}}_{\chi, x}^{-1}(b, w) - \underline{\mathbf{K}}^{-1}(b, w) = \frac{s}{2\pi} R_B \left(\frac{e^{i\nu(w)}}{w-x} \right) - \frac{s}{2\pi} \overline{R_B} \left(\frac{e^{-i\nu(w)}}{\overline{w-x}} \right) + O(|w-x|^{2s-2})$$

for $b \sim w$ (this corresponds to letting $y \rightarrow \infty$ in the previous estimate). Since the error term is summable, it follows that

$$\Re \sum_{(bw) \in E(\gamma_i)} \dot{K}_\gamma(b, w) \underline{\mathbf{K}}_{\chi, x}^{-1}(b, w) = -2s \log |x' - x| + c(x, \Lambda, \gamma, s) + O(|x - x'|^{2s-1})$$

Moreover, the left handside is unchanged if γ is another path started at x going through x' , as the difference can be written as the trace of a commutator. Thus $c(x, \Lambda, \gamma, s) = c(x, \Lambda, s)$. Consequently

$$\Re \text{Tr}(K_\gamma S_\gamma) = -2s \Re \log(y - x)^2 + c(x, \Lambda, s) + c(y, \Lambda, s) + o(1)$$

Lemma 23. *The kernel \underline{K}_χ^{-1} (with singularity at x) is local in the sense that $\underline{K}_\chi^{-1}(b, w)$ depends only on Λ restricted to a ball centered at x containing b, w .*

Proof. The discrete exponentials are by construction local. If w is adjacent to x , then $\underline{K}_\chi^{-1}(., w)$ has an integral representation in terms of discrete exponentials (in the variable b) and consequently $\underline{K}_\chi^{-1}(., w)$ is also local. For a general w , consider a simple path $(x_0 = x, x_1, \dots, x_n = x')$ from x to x' on M^\dagger , where x' is a face adjacent to w . Let $(w_i b_i) = (x_i x_{i+1})^\dagger$. By induction on the length of this path, it is easy to see that $\underline{K}_\chi^{-1}(., w)$ can be expressed as a linear combination of the $\underline{K}_{\chi, x_i}^{-1}(., w_i)$, which are local. Thus $\underline{K}_\chi^{-1}(., w)$ is itself local. \square

Consider the space $\{(\Lambda, x)\}$ of lozenge tilings rooted at a face (fixing the scale $\delta = 1$). As is customary for rooted graphs ([3]), we may define a distance by:

$$\text{dist}((\Lambda, x), (\Lambda', x')) = \inf\{\varepsilon > 0 : (\Lambda, x)|_{B(x, \varepsilon^{-1})} \text{ isomorphic to } (\Lambda', x')|_{B(x', \varepsilon^{-1})}\}$$

where graph isomorphism are required to preserve roots. We need to take into account embedding data, viz. lozenge angles. Thus we may refine the distance as follows:

$$\text{dist}((\Lambda, x), (\Lambda', x')) = \inf\{\varepsilon > 0 : (\Lambda, x)|_{B(x, \varepsilon^{-1})} \text{ isomorphic to } (\Lambda', x')|_{B(x', \varepsilon^{-1})} \text{ and corresponding angles differ by at most } \varepsilon\}$$

Then the set $\{(\Lambda, x) : \Lambda \text{ satisfies } (\spadesuit)\}$ is compact. Indeed, under the condition (\spadesuit) , degrees of vertices are bounded and the number of vertices in a subgraph is comparable to its area; thus there are finitely many isomorphism classes as rooted planar graphs for restrictions $(\Lambda, x)|_{B(x, \varepsilon^{-1})}$ (for fixed $\varepsilon > 0$). Each isomorphism class is parameterised by finitely many angles taking values in compact intervals.

Using locality of the kernels \underline{K}_χ^{-1} and their continuous dependence on lozenge angles for a given graph type, we conclude that $(\Lambda, x) \mapsto c(x, \Lambda, s)$ is a continuous function (for fixed s). In particular it is bounded under the condition (\spadesuit) for Λ .

Proposition 24. *If $s \in (0, \frac{1}{2})$, there are constants $c(x, \Lambda, s)$ such that*

$$|\langle \chi^{h(y)-h(x)} \rangle| \sim \exp(c(x, \Lambda, s) + c(y, \Lambda, s)) |x - y|^{-2s^2}$$

Proof. This is already established for a small $s_0 > 0$. We write

$$\log \left| \frac{\langle \chi^{h(y)-h(x)} \rangle}{\langle \chi_0^{h(y)-h(x)} \rangle} \right| = \int_{s_0}^s \Re \text{Tr}(K_\gamma K_\gamma^{-1}) du$$

and we have estimated the integrand. \square

Let us now show that $c(x, \Lambda, s) = c(\Lambda, s)$, which we already know for small s . We need only show

$$\left| \frac{\langle \chi^{h(y)-h(x')} \rangle}{\langle \chi^{h(y)-h(x)} \rangle} \right| \xrightarrow{y \rightarrow \infty} 1$$

if $x' \in M^\dagger$ is a neighbour of x ; in turn, this follows from

$$S_\rho(b, w) - \underline{K}_{\chi, x}^{-1}(b, w) = o(1)$$

as $y \rightarrow \infty$, where (bw) is the edge of M separating x from x' . From Lemma 15, estimates for S_ρ on the macroscopic ($|x - y|$) scale, and Lemma 16, we get

$$S_\rho(b, w) - \mathbf{K}_{\mathcal{X}, x}^{-1}(b, w) = O(|y - x|^{s-1+s+\eta})$$

for $\eta > 0$. This is enough to conclude that $c(x, \Lambda, s) = c(\Lambda, s)$.

We now discuss the “general” case:

$$\langle \exp(2i\pi \sum_{j=1}^n s_j (h(y_j) - h(x_j))) \rangle$$

where $s_1, \dots, s_n \in (0, \frac{1}{2})$. The estimates are as in the two point case, so we will simply record the needed changes in the computations. This problem is associated to a character ρ of $\pi_1(\mathbb{C} \setminus \{x_1, \dots, y_n\})$, and the kernel S_ρ inverting \mathbf{K} on sections of the associated line bundle. We start from the estimate for S_ρ (obtained from surgery and the two-point case):

$$S_\rho(z, w) = \frac{1}{2} R_B \left(e^{i\nu(w)} S(z, w) \right) + \frac{1}{2} \overline{R_B} \left(e^{-i\nu(w)} \bar{S}(z, w) \right) + o(R^{-1})$$

where all pairwise distances in $\{x_1, \dots, y_n, z, w\}$ are of order R , with

$$S(z, w) = \frac{1}{\pi(z-w)} \prod_{i=1}^n \left(\frac{(z-x_j)(w-y_j)}{(z-y_j)(w-x_j)} \right)^{s_j} = \frac{1}{\pi(z-w)} + r_\rho(w) + O(z-w)$$

$$\bar{S}(z, w) = \frac{1}{\pi(\bar{z}-\bar{w})} \prod_{i=1}^n \left(\frac{(\bar{z}-x_j)(\bar{w}-y_j)}{(\bar{z}-y_j)(\bar{w}-x_j)} \right)^{-s_j} = \frac{1}{\pi(\bar{z}-\bar{w})} + \bar{r}_\rho(w) + O(z-w)$$

where the estimate is for z, w away from the singularities, where the Robin kernels are given by

$$r_\rho(w) = \pi^{-1} \sum_{j=1}^n \left(\frac{s_j}{w-x_j} - \frac{s_j}{w-y_j} \right)$$

$$\bar{r}_\rho(w) = \pi^{-1} \sum_{j=1}^n \left(\frac{-s_j}{\bar{w}-x_j} - \frac{-s_j}{\bar{w}-y_j} \right) = -\overline{r_\rho(w)}$$

Let γ_j be a simple path from x_j to y_j (at macroscopic distance of the other singularities), and γ'_j the subpath from x'_j to y'_j . Since $r_\rho(w) \sim \frac{\pi^{-1}s_j}{w-x_j}$ near x_j and $r_\rho(w) \sim -\frac{\pi^{-1}s_j}{w-y_j}$ near y_j , we have

$$\oint_{\gamma'_j} r_\rho(w) dw = -\frac{s_j}{\pi} \log((y'_j - y_j)(x'_j - x_j)) + O(1)$$

Set

$$\oint_{\gamma_j}^{reg} r_\rho(w) dw = \lim_{x'_j \rightarrow x_j, y'_j \rightarrow y_j} \left(\oint_{\gamma'_j} r_\rho(w) dw + \frac{s_j}{\pi} \log((y'_j - y_j)(x'_j - x_j)) \right)$$

There is no determination issue for the real part of this regularised integral, which is what we need. The variational argument above shows that

$$\frac{\partial}{\partial s_j} \log |\langle \exp(2i\pi \sum_{j=1}^n s_j (h(y_j) - h(x_j))) \rangle| = -s_j \Re \oint_{\gamma_j}^{reg} \pi r_\rho(w) dw + s_j \Re \oint_{\gamma_j}^{reg} \pi \bar{r}_\rho(w) d\bar{w}$$

$$+ \partial_s c(x, \Lambda, s_j) + \partial_s c(y, \Lambda, s_j) + o(1)$$

and a direct computation yields

$$\oint_{\gamma_j}^{reg} \pi r_\rho(w) dw = s_j \log(-(x_j - y_j)^2) + \sum_{k \neq j} s_k \log \left(\frac{(y_j - x_k)(x_j - y_k)}{(y_j - y_k)(x_j - x_k)} \right)$$

as needed. We have obtained the following result on electric correlators:

Theorem 25. *For weights $s_1, \dots, s_{2n} \in (-\frac{1}{2}, \frac{1}{2})$, $s_1 + s_2 = \dots = s_{2n-1} + s_{2n} = 0$, and singularities $\{z_1, \dots, z_{2n}\}$ with pairwise distances of order R , we have as R goes to infinity:*

$$\left\langle \exp(2i\pi \sum_{j=1}^{2n} s_j h(z_j)) \right\rangle \sim \exp \left(\sum_{j=1}^{2n} c(\Lambda, |s_j|) \right) \prod_{1 \leq i < j \leq 2n} |z_i - z_j|^{2s_i s_j}$$

The constants $c(\Lambda, s)$ are bounded for Λ in the class (\spadesuit) and s in a compact subset of $(0, \frac{1}{2})$. Let us now discuss the dependence on Λ (or lack thereof). The locality and continuity arguments for $(\Lambda, x) \mapsto c(x, \Lambda, s) = c(\Lambda, s)$ show that, for Λ satisfying (\spadesuit) , $c(\Lambda, s)$ can be approximated within ε by looking at *any* ball of Λ of large enough radius $R(\varepsilon)$.

Consider Λ_θ a regular tiling of the plane by identical lozenges with angles $\theta, \pi - \theta$. One may glue Λ_θ and $\Lambda_{\theta'}$ along a line to obtain a lozenge tiling $\Lambda_{\theta, \theta'}$ (as in [21]). Then for small s ,

$$c(\Lambda_\theta, s) = c(\Lambda_{\theta, \theta'}, s) = c(\Lambda_{\theta'}, s)$$

by continuity in (Λ, x) .

More generally, consider the following condition for two rhombus tilings Λ_1, Λ_2 in the class (\spadesuit) : for some $\theta'_0 > 0$, there exists arbitrarily large balls $B_1 \subset \Lambda_1, B_2 \subset \Lambda_2$ and a rhombus tiling Λ_3 with angles $\geq \theta'_0$ containing copies of B_1 and B_2 . This generates an equivalence relation \sim on rhombus tilings; again by continuity we have (at least for small s): $c(\Lambda_1, s) = c(\Lambda_2, s)$ if $\Lambda_1 \sim \Lambda_2$.

The previous argument shows that $\Lambda_\theta \sim \Lambda_{\theta'}$. As another example, set Γ to be the triangular lattice and Λ_T the corresponding rhombus tiling; it contains an hexagonal lattice as a subgraph. Let Λ' be any rhombus tiling obtained from Λ_T by star-triangle transformations inside these hexagons. Then $\Lambda_T \sim \Lambda'$; we thus obtain an equivalence class with infinitely many elements which are distinct as graphs.

It is also easy to see that $\Lambda_T \sim \Lambda_{\pi/3}$, since one may glue a half-space of Λ_T and a half-space space $\Lambda_{\pi/3}$ to obtain a planar tiling Λ .

The triangular lattice has a two-parameter (up to isometry) family of embeddings with isometric faces, leading to a two-parameter family of rhombus tilings. In two steps (as in [21]), one sees that they all belong to the same equivalence class. Similarly, the isoradial embeddings of the square lattice (under (\spadesuit)), which are parameterised by two bi-infinite sequences of angles, fall in the same equivalence class.

It would be of some independent interest to decide whether \sim is trivial (coarse - and thus $c(\Lambda, s) = c(s)$) and if not describe its equivalence classes (building on the structure result of [32]).

8 Monomers and the Fisher-Stephenson conjecture

We now consider a planar graph M , derived from a lozenge tiling as illustrated in Figure 1, with “defects” consisting in a pair of missing vertices: a black vertex b_0 and a white vertex w_0 .

Specifically, consider a sequence $(\Xi_n)_{n \geq 1}$ of subgraphs of M bounded by a simple cycle on M^\dagger . We assume that for all n , b_0 and w_0 are in Ξ_n and that Ξ_n has a perfect matching; and that the inradius of Ξ_n , say seen from w_0 , goes to infinity as $n \rightarrow \infty$. Moreover we assume that $K_n^{-1}(b, w) \rightarrow \underline{K}^{-1}(b, w)$ for all

$b \in M_B, w \in M_W$ (here K_n denotes the restriction of K to Ξ_n); this is the case for “Temperleyan” boundary conditions ([26]). It follows that the sequence of the perfect matching measures on Ξ_n converges weakly to the measure on matchings on M described earlier.

We also assume that the limiting measure μ satisfies the following finite energy condition: $\mu\{(b_1 w_1) \in \mathbf{m}, \dots, (b_n, w_n) \in \mathbf{m}\} > 0$ for any finite subset $\{(b_i, w_i), 1 \leq i \leq n\}$ of edges of M than can be completed in a perfect matching of M .

Following Fisher-Stephenson [17], we consider the monomer correlation

$$\text{Mon}_{\Xi_n}(b_0, w_0) = \frac{\mathcal{Z}(\Xi_n \setminus \{b_0, w_0\})}{\mathcal{Z}(\Xi_n)}$$

where $\mathcal{Z}(\Xi)$ is the partition function of perfect matchings of the graph Ξ (with edge weight $|K(b, w)|$ for the edge (bw)).

In general, a Kasteleyn orientation of Ξ_n does not restrict to a Kasteleyn orientation of $\Xi'_n = \Xi_n \setminus \{b_0, w_0\}$. Indeed, if b_0 and w_0 are not neighbours in M , each correspond to a finite face of Ξ'_n where the clockwise odd condition is violated. In order to obtain a Kasteleyn orientation of Ξ'_n , one may introduce a “defect line” γ , i.e. a simple path γ on M^\dagger running from a face of M adjacent to b_0 to a face adjacent to w_0 . Reversing orientations of all edges crossing γ yields a Kasteleyn orientation of Ξ'_n . Thus, let us fix such a defect line γ and define $K' : \mathbb{R}^{M_B \setminus \{b_0\}} \rightarrow \mathbb{R}^{M_W \setminus \{w_0\}}$ by $K'(w, b) = -K(w, b)$ if γ crosses (bw) and $K(w, b)$ otherwise. We denote by K_n, K'_n the restrictions of K, K' to Ξ_n, Ξ'_n respectively (throughout, the prime recalls the sign change across the defect line). Then [17]:

$$\text{Mon}_{\Xi_n}(b_0, w_0) = \frac{\mathcal{Z}(\Xi'_n)}{\mathcal{Z}(\Xi_n)} = \pm \frac{\det(K'_n)}{\det(K_n)}$$

In order to compare K'_n, K_n , it is rather convenient to extend K'_n to an operator $\mathbb{R}^{\Xi_B} \rightarrow \mathbb{R}^{\Xi_W}$ by setting $K'(w_0, b_0) = 1$, with all other new matrices elements $K'(w_0, b), K'(w, b_0)$ vanishing (one may think of an edge “handle” connecting directly b_0, w_0). This does not change the determinant of K'_n (up to sign). Then

$$\text{Mon}_{\Xi_n}(b_0, w_0) = \pm \det(K'_n K_n^{-1})$$

Clearly $K'_n - K_n$ has bounded support (vertices adjacent to γ), and does not depend on n (for n large enough) on its support. It follows from the assumption of pointwise convergence of K_n^{-1} that:

$$\text{Mon}_M(b_0, w_0) = \left| \lim_{n \rightarrow \infty} \det(K'_n K_n^{-1}) \right| = |\det(K' \underline{K}^{-1})|$$

This defines the infinite volume monomer correlation $\text{Mon}_M(b_0, w_0)$. Under the finite energy assumption, $\text{Mon}_M(b_0, w_0)$ is positive. Let us sketch a direct argument. Consider a simple path from b to w on M : $(b_0 = b, w_0, b_1, \dots, b_n, w_n = w)$. Then matchings of Ξ'_n containing $(w_0, b_1), \dots, (w_{n-1}, b_n)$ are in bijection with matchings of Ξ_n containing $(b_0, w_0), \dots, (b_n, w_{n+1})$ (see [25] for related arguments), which gives the lower bound:

$$\text{Mon}_M(b, w) \geq \frac{|K(w_0, b_1) \dots K(w_{n-1}, b_n)|}{|K(w_0, b_0) \dots K(w_n, b_n)|} \mu\{(b_0, w_0) \in \mathbf{m}, \dots, (b_n, w_n) \in \mathbf{m}\}$$

One may choose the path so that (b_0, \dots, b_{n+1}) is a simple path on Γ and b_{n+1} is connected to infinity in $\Gamma \setminus \{b_0, \dots, b_n\}$. Then

$$\mu\{(b_0, w_0) \in \mathbf{m}, \dots, (b_n, w_n) \in \mathbf{m}\} = |K(w_0, b_0) \dots K(w_n, b_n) (\det \underline{K}^{-1}(b_i, w_j))_{0 \leq i, j \leq n}|$$

This is positive; indeed, otherwise one could find a non trivial $f \in (\mathbb{C}^{M_B})$ vanishing at infinity such that Kf is supported on $\{w_0, \dots, w_n\}$, and vanishes at $\{b_0, \dots, b_n\}$. It is then easy to see that $f|_\Gamma$ is discrete harmonic

except at $\{b_0, \dots, b_{n+1}\}$. Since the harmonic measure of b_{n+1} seen from infinity in $\Gamma \setminus \{b_0, \dots, b_{n+1}\}$ is positive, this forces $f(b_{n+1}) = 0$, then $f|_\Gamma = 0$, then $f = 0$, yielding a contradiction.

The monomer correlation question bears on the asymptotic behaviour of $\text{Mon}_M(b_0, w_0)$ as $|b_0 - w_0| \rightarrow \infty$, eg with w_0 fixed. The Fisher-Stephenson conjecture [17] proposes:

$$\text{Mon}_{\mathbb{Z}^2}(b, w) \sim c|b - w|^{-\frac{1}{2}}$$

In [25], Kenyon analyses the case of a white defect in the bulk and a black defect on the boundary of a simply connected portion of the square lattice. This leads to the length exponent for the loop-erased random walk.

More generally, if $\{b_1, \dots, b_n\}$ and $\{w_1, \dots, w_n\}$ are n -tuples of black and white vertices of M , one may consider the $2n$ -point monomer correlation:

$$\text{Mon}_M(b_1, \dots, b_n; w_1, \dots, w_n) = \lim_{n \rightarrow \infty} \frac{\mathcal{Z}(\Xi_n \setminus \{b_1, \dots, b_n, w_1, \dots, w_n\})}{\mathcal{Z}(\Xi_n)}$$

Let us give a heuristic interpretation of these monomer correlators. If we apply the construction of the height function to dimer-monomer configurations on M , with monomers located at the fixed $\{b_1, \dots, b_n, w_1, \dots, w_n\}$, we obtain an additively multivalued function on M^\dagger , which increases by 1 (resp. -1) when cycling counter-clockwise around a white (resp. black) vertex. In the absence of defects, the height field converges to a free field (in the plane). Interpreting the monomers as discrete versions of magnetic operators for the free field, one may expect

$$\text{Mon}_M(b_1, \dots, b_n; w_1, \dots, w_n) \sim c \langle : \mathcal{O}_1(w_1) \dots \mathcal{O}_1(w_n) \mathcal{O}_{-1}(b_1) \dots \mathcal{O}_{-1}(b_n) : \rangle_{\mathbb{C}}$$

in some asymptotic regime, for instance as the pairwise distance between insertions goes to infinity (alternatively as the lattice mesh goes to zero). For the planar free field, the magnetic correlator is explicitly

$$\langle : \prod_j \mathcal{O}_{\alpha_j}(z_j) : \rangle_{\mathbb{C}} = \prod_{j < k} |z_j - z_k|^{\frac{g-1}{2\pi} \alpha_j \alpha_k}$$

and here $g = \pi^{-1}$, in agreement with eg Corollary 4. This yields the natural extension of the Fisher-Stephenson conjecture:

$$\text{Mon}_M(z_1, \dots, z_{2n}) \sim c \prod_{i < j} |z_i - z_j|^{\varepsilon_i \varepsilon_j / 2}$$

where $\varepsilon_i = 1$ (resp. -1) if z_i is black (resp. white) and $\sum_i \varepsilon_i = 0$. This is supported in particular by Ciucu's work (see [9] and references therein).

The (somewhat stronger) variational form of the statement we shall derive here consists in estimating $\log(\text{Mon}_M(b, w) / \text{Mon}_M(b', w))$ within $O(|b - w|^{-1-\varepsilon})$ for b', b black vertices on the same face of M (or similar quantities for higher order correlations).

The relevant family of holomorphic line bundles are the line bundles over $\Sigma = \hat{\mathbb{C}} \setminus \{z_1, \dots, z_{2n}\}$ with holomorphic sections in U of type:

$$s_U(z) = \prod_{i=1}^{2n} (z - z_i)^{\varepsilon_i / 2} g(z)$$

where g is holomorphic in $U \setminus \hat{\mathbb{C}}$ and vanishing at infinity.

8.1 Monomer pairs

In this subsection, we focus on monomer pair correlations, in the framework laid out at the start of the section.

Let us observe that one may define a dimer measure $\mu_{M'}$ on matchings of $M' = M \setminus \{b_0, w_0\}$ as the weak limit of the (weighted) dimer measures on Ξ'_n as $n \rightarrow \infty$. It suffices ([24]) to observe that $(K'_n)^{-1} = K_n^{-1}(K'_n K_n^{-1})^{-1}$, and as $n \rightarrow \infty$, K_n^{-1} converges pointwise to \underline{K}^{-1} and $(K'_n K_n^{-1})$ converges pointwise to an invertible (since $\text{Mon}(b_0, w_0) > 0$), bounded support perturbation of the identity. Let us denote $(K')^{-1}$ the limiting kernel (or $(K'_{b_0, w_0})^{-1}$ if we need to emphasise the position of monomers). It satisfies $K'(K')^{-1}(\cdot, w) = \delta_w$ for all $w \in M_W \setminus \{w_0\}$, and $(K')^{-1}(b, w) = O(|b - w|^{-1})$.

Denote by (b_0, w_1, b_1) three consecutive vertices (in counterclockwise order, say) on a face of M . We have:

$$\begin{aligned} \mathcal{Z}(\Xi_n \setminus \{b_0, w_1, b_1, w_0\}) &= \mathcal{Z}(\Xi_n \setminus \{b_0, w_0\}) |K(w_1, b_1)|^{-1} \mu_{\Xi_n \setminus \{b_0, w_0\}} \{(b_1, w_1) \in \mathfrak{m}\} \\ &= \mathcal{Z}(\Xi_n \setminus \{b_1, w_0\}) |K(w_1, b_0)|^{-1} \mu_{\Xi_n \setminus \{b_1, w_0\}} \{(b_0, w_1) \in \mathfrak{m}\} \end{aligned}$$

where we consider two defects: the monomer w_0 , and the “trimer” $\{b_0, w_1, b_1\}$. Consequently,

$$\frac{\text{Mon}_{\Xi_n}(b_1, w_0)}{\text{Mon}_{\Xi_n}(b_0, w_0)} = \left| \frac{K(w_1, b_0)}{K(w_1, b_1)} \right| \frac{\mu_{\Xi_n \setminus \{b_0, w_0\}} \{(b_1, w_1) \in \mathfrak{m}\}}{\mu_{\Xi_n \setminus \{b_1, w_0\}} \{(b_0, w_1) \in \mathfrak{m}\}} = \pm \frac{(K'_{\Xi_n \setminus \{b_0, w_0\}})^{-1}(b_1, w_1)}{(K'_{\Xi_n \setminus \{b_1, w_0\}})^{-1}(b_0, w_1)}$$

and taking the limit as $n \rightarrow \infty$,

$$\frac{\text{Mon}_M(b_1, w_0)}{\text{Mon}_M(b_0, w_0)} = \pm \frac{(K'_{b_0, w_0})^{-1}(b_1, w_1)}{(K'_{b_1, w_0})^{-1}(b_0, w_1)}$$

Thus we simply need to estimate $S_{b_0, w_0} = (K'_{b_0, w_0})^{-1}$ precisely enough near the singularity b_0 . The argument is at times somewhat technical; let us briefly sketch the line of reasoning.

1. Random walk arguments give an estimate on \hat{S}_ρ , an inverting kernel for K'_{b_0, w_0} which is bounded (rather than vanishing) at infinity.
2. Bounded functions in the kernel of K'_{b_0, w_0} are classified.
3. The correct inverting kernel S_{b_0, w_0} is constructed, and estimated in the macroscopic scale.
4. A priori estimates for S_{b_0, w_0} when one or both arguments are near singularities are given.
5. \hat{g} , an approximation of $S_{b_0, w_0}(\cdot, w_1)$ is constructed, using as building blocks discrete holomorphic functions mesoscopically near singularities, and continuous holomorphic functions elsewhere.
6. $\hat{g}(b_1) - S_{b_0, w_0}(\cdot, w_1)$ is estimated within $O(|b_0 - w_0|^{-1-\varepsilon})$ (this requires controlling the leading term, which depends only on the local geometry around x , and the first subleading correction which carries the “global” information).

Recall that K'_{b_0, w_0} is obtained from K by deleting the row and column corresponding to the pair of monomers and changing the sign of matrix entries corresponding to edges crossing the defect line γ which runs from $x \in M^\dagger$ adjacent to b_0 to $y \in M^\dagger$ adjacent to w_0 :

$$K'_{b_0, w_0} : \mathbb{R}^{M_B \setminus \{b_0\}} \longrightarrow \mathbb{R}^{M_W \setminus \{w_0\}}$$

Displacing γ (with endpoints fixed) results in composing K_{b_0, w_0} with diagonal matrices (with ± 1 diagonal coefficients). Equivalently, one may consider multivalued functions on M corresponding to the character $\rho : \pi_1(\mathbb{C} \setminus \{x, y\}) \rightarrow \{\pm 1\}$ with monodromy -1 around x, y . One may identify elements of $\mathbb{R}^{M_B \setminus \{b_0\}}$ with functions on M_B vanishing at b_0 , and elements of $\mathbb{R}^{M_W \setminus \{w_0\}}$ as functions on M_W modulo δ_{w_0} .

We start with by constructing an inverting kernel S_{b_0, w_0} for K'_{b_0, w_0} , at least for $|b_0 - w_0|$ large enough.

Assume without loss of generality that $b_0 \in \Gamma^\dagger$. We observe that the kernel \hat{S}_ρ obtained from the Green kernel for the random walk with monodromy at b_0, w_0 on Γ vanishes at b_0 (see Lemma 17) and afterwards). Hence $\mathbf{K}'_{b_0, w_0} \hat{S}_\rho(\cdot, w) = \delta_w$. Moreover we have

$$\hat{S}_\rho(z, w) = \frac{1}{2} R_B \left(\sqrt{\frac{(z-x)(z-y)}{(w-x)(w-y)}} \cdot \frac{e^{i\nu(w)}}{\pi(z-w)} \right) + (cc) + o(1/|x-y|)$$

for (z, w) in a compact set of $\{z \neq x, y, w; w \neq x, y, z\}$. Note that in this case, a closed form expression for the continuous Green kernel G_ρ is easily obtained by going to a double cover of $\hat{\mathbb{C}} \setminus \{b_0, w_0\}$.

The kernel \hat{S}_ρ does not vanish at infinity; on the other hand we have some freedom since $S_{b_0, w_0}(\cdot, w)$ is not required to be holomorphic at w_0 . In order to correct \hat{S} to obtain an inverting kernel vanishing at infinity, we need to study bounded functions in the kernel of \mathbf{K}'_{b_0, w_0} (this is a point where the argument differs from the construction for electric correlators).

We claim that $\dim\{f : \mathbf{K}'_{b_0, w_0} f = 0, \|f\|_\infty < \infty\} \leq 2$. Indeed, $\mathbf{K}'_{b_0, w_0} f = 0$ implies $\Delta_\Gamma f|_\Gamma$ is supported on the two vertices of Γ abutting w_0 , and there are no nonzero bounded ρ -multivalued harmonic functions on Γ ($f|_{\Gamma^\dagger}$ is specified by harmonic conjugation and $f(b_0) = 0$). We simply need to construct two linearly independent bounded functions in the kernel of \mathbf{K}'_{b_0, w_0} .

We have constructed (Lemma 11) g such that $\mathbf{K}_{-1, y} g = 0$ outside of w_0 and

$$g(z) = R_B((z-y)^{-1/2}) + O(|z-y|^{-3/2})$$

We then consider $f = g\mathbf{1}_B - \hat{S}_\rho(g\mathbf{1}_B)$ where $B = B(y, r)$, $r \leq \frac{1}{2}|x-y|$. Plainly, f is bounded and $\mathbf{K}_{b_0, w_0} f = 0$. Since

$$\frac{1}{2i\pi} \oint_{C(y, r)} \frac{1}{\sqrt{w-y}} \cdot \sqrt{\frac{(z-x)(z-y)}{(w-x)(w-y)}} \cdot \frac{dw}{z-w} = \sqrt{\frac{z-x}{(z-y)(y-x)}} - \frac{\mathbf{1}_B(z)}{\sqrt{z-y}}$$

and the contribution from the conjugate term vanishes, we get

$$f(z) = R_B \left(\sqrt{\frac{z-x}{(z-y)(y-x)}} \right) + o(|x-y|^{-1/2})$$

for z in a compact subset of $\mathbb{C} \setminus \{x, y\}$. We know a priori that $f|_\Gamma$ (resp. $f|_{\Gamma^\dagger}$) has a limit as $z \rightarrow \infty$. Plainly, $\lim_{z \rightarrow \infty} f|_\Gamma(z) = 1/\sqrt{y-x} + o(|x-y|^{-1/2})$ and $\lim_{z \rightarrow \infty} f|_{\Gamma^\dagger}(z) = i/\sqrt{y-x} + o(|x-y|^{-1/2})$.

We observe that \bar{f} is also in the kernel of \mathbf{K}'_{b_0, w_0} (a real operator); examining the limits of $f|_\Gamma$, $f|_{\Gamma^\dagger}$ at infinity shows that f, \bar{f} are linearly independent. Moreover, there is a unique kernel S_{b_0, w_0} inverting \mathbf{K}'_{b_0, w_0} and vanishing at infinity. By uniqueness, this kernel is real. It may be realised as

$$S_{b_0, w_0}(b, w) = \hat{S}_\rho(b, w) - \Re(a(w)f(b))$$

where $a(w)$ is fixed by the behaviour as $b \rightarrow \infty$: $a(w) = \frac{e^{i\nu(w)}}{2} \sqrt{\frac{y-x}{(w-x)(w-y)}} + o(1/|x-y|^{1/2})$. Consequently, we obtain the estimate

$$S_{b_0, w_0}(z, w) = \frac{1}{2} R_B \left(\sqrt{\frac{(z-x)(w-y)}{(z-y)(w-x)}} \cdot \frac{e^{i\nu(w)}}{\pi(z-w)} \right) + (cc) + o(1/|x-y|)$$

for (z, w) in a compact set of $\{z \neq x, y, w; w \neq x, y, z\}$.

Let us denote by (b_0, w_1, b_1, w_2) the vertices of the face x of M , listed counterclockwise. We want to estimate $S_{b_0, w_0}(w_1, b_1)$ within $O(|x-y|^{-1-\varepsilon})$. We shall need basic discrete holomorphic functions with

monodromy $\chi = -1$ around the singularity $x \in M^\dagger$. Set $g = \mathbf{K}_\chi^{-1}(\cdot, w_1)$, $f_{-1} = \Re f_\chi$ at $\chi = -1$; $h = \overline{f_{1,-1}}/\tau$ where $\tau = (b_1 - x)/|b_1 - x|$. We list some useful evaluations (Lemmas 10, 11).

$$\begin{aligned} g(b) &= \frac{\Gamma(\frac{1}{2})}{2} R_B \left(\frac{e^{i\nu(w_1)}}{\pi(b-w)} \sqrt{\frac{b-x}{w_1-x}} \right) + O(|b-x|^{-3/2}) \\ g(b_0) &= \frac{e^{i\theta_0} - 1}{2 \sin \theta_0} \\ g(b_1) &= \frac{e^{i\theta_1} - 1}{2 \sin \theta_1} \\ h(b) &= \sqrt{2} \Gamma(-\frac{1}{2}) R_B(\sqrt{b-x}) + O(|b-x|^{-3/2}) \\ h(b_0) &= -i\pi(2(b_0-x))^{1/2} \\ h(b_1) &= -\pi(2(b_1-x))^{1/2} \end{aligned}$$

where $\theta_i = \arg(\frac{b_i - w_1}{x - w_1})$, and we assume that the defect line γ does not cross $(b_0 w_1)$, $(b_1 w_1)$.

Let us review some a priori estimates on $S_{b_0, w_0}(b, w)$ for b close to x , and w close to x, y, ∞ respectively.

Lemma 26. *There is $\varepsilon > 0$ such that:*

1. *If $\mathbf{K}_{-1,x}f = 0$ in $B(x, r)$, $f(b_0) = 0$, then $f(b) = O((|b-x|/r)^\varepsilon \|f_{\partial B(x,r)}\|_\infty)$ for $b \in B(x, r)$.*
2. *There is a unique kernel S_{b_0} s.t. $S_{b_0}(\cdot, w)$ has monodromy (-1) around x , $\mathbf{K}_{-1}S_{b_0}(\cdot, w) = \delta_w$, and $S_{b_0}(\cdot, w)$ vanishes at b_0 and infinity.*
3. *$S_{b_0, w_0}(b, w) = O(|w-x|^{-1-\varepsilon})$ if b adjacent to x , $|w-x| \leq \frac{1}{2}|x-y|$*
4. *$S_{b_0, w_0}(b, w) = O(|w-y|^{1/2-\varepsilon}|x-y|^{-3/2})$ if b adjacent to x , $|w-y| \leq \frac{1}{2}|x-y|$*
5. *$S_{b_0, w_0}(b, w) = O(|w-x|^{-1}|x-y|^{-\varepsilon})$ if b adjacent to x , $|w-x| \geq \frac{1}{2}|x-y|, |w-y| \geq \frac{1}{2}|x-y|$.*

Proof. 1. This follows from observing that $f|_\Gamma$ is harmonic, with monodromy (-1) around x , and writing $f|_{\Gamma^\dagger}$ as the harmonic conjugate of $f|_\Gamma$ vanishing at b_0 .

2. Uniqueness follows from the fact that the space of (-1) -multivalued bounded holomorphic functions is spanned by a single function f_{-1} which does not vanish at b_0 . By uniqueness this kernel is real. As before (when w_0 is at finite distance), one can write $S_{b_0}(\cdot, w)|_\Gamma$ as a first difference of the Green kernel on Γ (with monodromy -1 around x) and extend it to Γ^\dagger by harmonic conjugation. This leads to the a priori estimate

$$S_{b_0}(b, w) = O(|w|^{-1}((|b|/|w|)^\varepsilon \wedge (|w|/|b|)^\varepsilon))$$

eg if $|b-w| \geq \frac{1}{2}|w|$ (setting $x = 0$ for notational simplicity).

3. Let us write

$$S_{b_0, w_0}(\cdot, w) = \mathbf{1}_B S_{b_0}(\cdot, w) - S_{b_0, w_0}(\mathbf{K}'_{b_0, w_0}(\mathbf{1}_B S_{b_0}(\cdot, w)))$$

where $B = B(x, \frac{3}{4}|x-y|)$. We have $\mathbf{K}'_{b_0, w_0}(\mathbf{1}_B S_{b_0}(\cdot, w))(w') = O(|w|^{-1}(|w|/R)^\varepsilon)$ for $w' \in \partial B$ (where $R = |x-y|$), and $S_{b_0, w_0}(b, w') = O(R^{-1-\varepsilon})$ by 1. Thus $S_{b_0, w_0}(b, w) = O(|w|^{-1-\varepsilon})$.

4. Here the reference kernel S_y is constructed from the Green kernel which has monodromy (-1) at y and is such that $S_y(b, w) = O(|w-y|^{-1})$ if $|b-y|$ and $|b-w|$ are comparable to $|w-y|$. We can construct two functions f_y, \bar{f}_y with monodromy (-1) at y which are holomorphic except possibly at w_0 and with expansion $\bar{f}_y = R_B((b-w_0)^{-1/2}) + O((b-w_0)^{-3/2})$. Correcting $S_y(\cdot, w)$ by an appropriate linear combination of f_y, \bar{f}_y , we obtain a kernel \hat{S}_y such that $\hat{S}_y(b, w) = O(|w-y|^{-1}(|b-y|/|w-y|)^{-3/2+\varepsilon})$

for $|b - y| \geq 2|w - y|$, say.

Then we may represent $S_{b_0, w_0}(\cdot, w)$ as

$$S_{b_0, w_0}(\cdot, w) = \mathbf{1}_B \hat{S}_y(\cdot, w) - S_{b_0, w_0}(\mathbf{1}_B \hat{S}_y(\cdot, w))$$

with $B = B(y, \frac{1}{2}|x - y|)$ to obtain for b close to x :

$$S_{b_0, w_0}(b, w) = O(|x - y|^{-\varepsilon} |w - y|^{-1} (|w - y|/|x - y|)^{3/2 - \varepsilon})$$

5. We simply write

$$S_{b_0, w_0}(\cdot, w) = \mathbf{1}_B \hat{\mathbf{K}}^{-1}(\cdot, w) - S_{b_0, w_0}(\mathbf{1}_B \hat{\mathbf{K}}^{-1}(\cdot, w))$$

with $B = B(x, \frac{1}{2}|x - y|) \cup B(y, \frac{1}{2}|x - y|)$.

□

We now want to construct an approximation of $S_{b_0, w_0}(\cdot, w_1)$. We are looking for such a function \hat{g} of the form: \hat{g} is a linear combination of g, \bar{g}, h, \bar{h} in $B(x, r)$;

$$\hat{g}(b) = \Re R_B \left(\frac{\mu}{\pi(b - w_1)} \sqrt{\frac{(b - b_0)(w_1 - w_0)}{(w_1 - b_0)(b - w_0)}} \right)$$

in $(B(x, r) \cup B(y, \tilde{r}))^c$. In $B(y, \tilde{r})$ we set $\hat{g} = \Re(\gamma f_y)$ where f_y is a function such that $f_y(b) = R_B((b - y)^{-1/2}) + O(|b - y|^{-3/2})$, and f_y is (-1) -multivalued around y and is holomorphic except at w_0 . Set $R = |x - y|$.

We need: $\hat{g}(b_0) = 0$, $(\mathbf{K}'_{b_0, w_0} \hat{g})(w_1) = 1$, and the values of \hat{g} across $\partial B(x, r)$ differ by $o(\sqrt{r}/R)$: this defines uniquely \hat{g} as a linear combination of g, \bar{g}, h, \bar{h} and fixes μ . Since $\bar{\hat{g}}$ satisfies the same conditions, we may set $\hat{g} = \Re(\alpha g + \beta h)$ in $B(x, r)$. We get $\mu = \frac{\alpha}{2} e^{i\nu(w_1)} \Gamma(1/2) \sqrt{\frac{w_1 - b_0}{w_1 - x}}$.

Then $\hat{g} - S_{b_0, w_0}(\mathbf{K}'_{b_0, w_0} \hat{g} - \delta_{w_1}) = S_{b_0, w_0}(\cdot, w_1)$. We need to fit α, β in order to make the error term small near b_0 . Since

$$\begin{aligned} \frac{1}{b - w_1} \sqrt{\frac{(b - b_0)(w_1 - w_0)}{(w_1 - b_0)(b - w_0)}} &= \sqrt{\frac{b - b_0}{w_1 - b_0}} \left(\frac{1}{b - w_1} + \frac{1}{2} \cdot \frac{1}{w_0 - b} + O(|b - w_1|/|b - w_0|^2) \right) \\ &= \frac{1}{\sqrt{(w_1 - b_0)(b - w_0)}} (1 + O(|b - w_0|/R)) \end{aligned}$$

we require: $\Re(\alpha) = 1$, $\Re(\alpha g(b_0) + \beta h(b_0)) = 0$ and

$$\beta \sqrt{2} \Gamma(-\frac{1}{2}) = \alpha \frac{\Gamma(\frac{1}{2}) e^{i\nu(w_1)}}{2\pi \sqrt{w_1 - x}} \cdot \frac{1}{2(w_0 - b_0)}$$

or $\beta = \alpha e^{i\nu(w_1)} / (8\pi \sqrt{2(w_1 - x)(b_0 - w_0)}) = \alpha z$ with $z = O(1/R)$. Let us remark that $\Im(g(b_i)) = \frac{1}{2}$ does not depend on the local geometry (while $\Re(g(b_i))$ does). From $\Re(\alpha g(b_0) + \beta h(b_0)) = 0$, we get

$$\Re(g(b_0) + \beta h(b_0)) = \frac{1}{2} \Im(\alpha)$$

and thus $\alpha = 1 + 2i\Re(g(b_0)) = 2i\overline{g(b_0)} + O(1/R)$, and by iteration:

$$\alpha = 2i\overline{g(b_0)} + 4i\Re(i\overline{g(b_0)}zh(b_0)) + O(1/R^2)$$

and we have

$$\begin{aligned} \hat{g}(b_1) &= \Re(\alpha g(b_1) + \beta h(b_1)) = 2\Re(i\overline{g(b_0)}g(b_1)) + 4\Re(i\overline{g(b_0)}zh(b_0))(-\frac{1}{2}) + \Re(2i\overline{g(b_0)}zh(b_1)) + O(1/R^2) \\ &= 2\Re(i\overline{g(b_0)}g(b_1)) + \Re(2i\overline{g(b_0)}z(h(b_1) - h(b_0))) \end{aligned}$$

Then $K'_{b_0, w_0} \hat{g} = O(r^{-1/2}/R + r^{-3/2} + r^{3/2}/R^2) = O(\sqrt{r}/R + r^{-3/2})$ on $\partial B(x, r)$. This contributes to a bound of order $O(r^{-1/2-\varepsilon}R^{-1} + r^{-3/2-\varepsilon} + r^{3/2-\varepsilon}R^{-2})$ on the estimation error $(\hat{g}(b) - S_{b_0, w_0}(b, w_1))$ for b within $O(1)$ of x .

In $B(x, R) \setminus B(x, r)$, we have $K'_{b_0, w_0} \hat{g} = O(|b-x|^{-1/2-3})$, contributing to an error of order $O(\sum_{k=r}^R k^{-1/2-3-\varepsilon}) = O(r^{-1/2-2-\varepsilon})$. In $B(y, R) \setminus B(y, \tilde{r})$, we have $(K'_{b_0, w_0} \hat{g})(w) = O(|w - w_0|^{-1/2-3})$, contributing to an error of order $O(\sum_{k=\tilde{r}}^R k^{-1/2-3} k^{3/2-\varepsilon} R^{-3/2}) = O(\tilde{r}^{-1-\varepsilon} R^{-3/2})$. On $\partial B(y, \tilde{r})$, we estimate $K'_{b_0, w_0} \hat{g} = O(\tilde{r}^{-3/2} + \tilde{r}^{1/2}/R)$, leading to an error $O((\tilde{r}^{-3/2} + \tilde{r}^{1/2}/R) \tilde{r}^{3/2-\varepsilon} R^{-3/2})$. Finally, outside of $B(x, R) \cup B(y, R)$ we have $(K'_{b_0, w_0} \hat{g})(w) = O(|w|^{-4} \sqrt{R})$ and an associated error of order $O(\sum_{k=R}^{\infty} k^{-4} \sqrt{R} R^{-\varepsilon}) = O(R^{1/2-\varepsilon-3})$. Summing up, by setting $r = R^{2/3}$, $\tilde{r} = o(R^{3/4})$, we obtain:

$$S_{b_0, w_0}(b_1, w_1) = \hat{g}(b_1) + O(|b_0 - w_0|^{-1-\varepsilon}).$$

We are interested in exchanging the roles of b_0, b_1 . We have

$$S_{b_0, w_0}(b_1, w_1) = 2\Re(\overline{ig(b_0)}g(b_1)) + \Re\left(2\overline{ig(b_0)}z(h(b_1) - h(b_0))\right) + O(R^{-1-\varepsilon})$$

$$S_{b_1, w_0}(b_0, w_1) = 2\Re(\overline{ig(b_1)}g(b_0)) + \Re\left(2\overline{ig(b_1)}z(h(b_0) - h(b_1))\right) + O(R^{-1-\varepsilon})$$

Observe that $2\Re(\overline{ig(b_0)}g(b_1)) = g(b_1) - g(b_0)$, which is real (since $\Im(g(b_0)) = \Im(g(b_1)) = \frac{1}{2}$), and that $h(b_1) - h(b_0) = -2\pi(2(b_1 - x))^{1/2}$. An examination of the local geometry yields $ie^{i\nu(w_1)} \frac{h(b_1) - h(b_0)}{4\pi\sqrt{2(w_1 - x)}} = -\frac{1}{2}(b_1 - b_0)$. Consequently,

$$\begin{aligned} \frac{\text{Mon}_M(b_1, w_0)}{\text{Mon}_M(b_0, w_0)} &= -\frac{S_{b_0, w_0}(b_1, w_1)}{S_{b_1, w_0}(b_0, w_1)} = 1 + \Re(2iz(h(b_1) - h(b_0))) + O(|b_0 - w_0|^{-1-\varepsilon}) \\ &= 1 - \frac{1}{2}\Re\left(\frac{b_1 - b_0}{b_0 - w_0}\right) + O(|b_0 - w_0|^{-1-\varepsilon}) \end{aligned}$$

and thus we have obtained:

Proposition 27. *There is $c(\Lambda, w) > 0$ such that*

$$\text{Mon}_M(b, w) \sim c(\Lambda, w)|b - w|^{-1/2}$$

as $b \rightarrow \infty$.

Moreover c is a continuous function of the face-rooted rhombus tiling (Λ, w) , and is bounded away from 0 and ∞ on (\spadesuit) . Presumably, c depends only on Λ (and thus is constant on a large class of rhombus tiling, see the discussion after Theorem 25). To see this, it is enough to check that

$$\frac{\text{Mon}(b, w')}{\text{Mon}(b, w)} \rightarrow 1$$

as $b \rightarrow \infty$, if w, b', w' are consecutive vertices on the boundary of a face of M . In turn this is equivalent to

$$\left| \frac{S_{b, w}(b', w')}{S_{b, w'}(b', w)} \right| \rightarrow 1$$

as $b \rightarrow \infty$. This seems to require another *ad hoc* local computation, which we leave to the dedicated reader.

Notice however that if M is the square (or rectangular) lattice, one can interchange the roles of black and white vertices and consequently $c(\mathbb{Z}^2, w) = c(\mathbb{Z}^2)$.

8.2 General monomer correlators

In this subsection we indicate how to extend the previous argument to the general case of $2m$ monomers $b_1, \dots, b_m, w_1, \dots, w_m$ in the plane, which does not involve any substantial additional difficulty. We denote $\underline{b} = \{b_1, \dots, b_m\}$, $\underline{w} = \{w_1, \dots, w_m\}$.

Let x_i (resp. y_i) be the face of M adjacent to $b_i \in M_B$ (resp. $w_i \in M_W$). Let γ_i be a defect line running from x_i to y_i on M^\dagger ; we assume that the γ_i 's are disjoint and that the pairwise distances between singularities are of order $R \gg 1$. Let

$$K' : \mathbb{R}^{M_B \setminus \{b_1, \dots, b_m\}} \longrightarrow \mathbb{R}^{M_W \setminus \{w_1, \dots, w_m\}}$$

be the operator obtained from K' by removing rows and columns corresponding to monomers and changing signs of entries corresponding to edges of M crossing one of the defect lines. Equivalently, one may think of K' as mapping $\{f \in \mathbb{R}^{M_B} : f(b_1) = \dots = f(b_m) = 0\}$ to $\mathbb{R}^{M_W} / \mathbb{R}^{\{w_1, \dots, w_m\}}$.

By surgery (Lemma 8) and induction on m (as we did for multiple electric correlators), we get that for R large enough, there exists a unique inverting kernel $S_{\underline{b}, \underline{w}}$ for K' vanishing at infinity and that

$$S_{\underline{b}, \underline{w}}(z, w) = \frac{1}{2} R_B \left(e^{i\nu(w)} S(z, w) \right) + (cc) + o(1)$$

where all pairwise distances are of order R and S is the continuous kernel

$$S(z, w) = \prod_{i=1}^m \left(\frac{(z - x_i)(w - y_i)}{(z - y_i)(w - x_i)} \right)^{1/2} \cdot \frac{1}{\pi(z - w)}$$

The corresponding Robin kernel is

$$r(w) = \lim_{z \rightarrow w} \left(S(z, w) - \frac{1}{\pi(z - w)} \right) = \frac{1}{2\pi} \sum_{i=1}^m \left(\frac{1}{w - x_i} - \frac{1}{w - y_i} \right)$$

so that

$$\hat{r}(x_k) = \lim_{x \rightarrow x_k} \left(r(w) - \frac{1}{2\pi(w - x_k)} \right) = \frac{1}{2\pi(y_k - x_k)} + \frac{1}{2\pi} \sum_{i \neq k} \left(\frac{1}{x_k - x_i} - \frac{1}{x_k - y_i} \right)$$

If (b_0, w, b'_0) are on the boundary of the face x_k (with $b_0 = b_k$), a straightforward extension of the two-point argument yields

$$\begin{aligned} S_{\underline{b}, \underline{w}}(b'_0, w) &= 2\Re(\overline{ig(b_0)}g(b'_0)) + \Re \left(\frac{\overline{ig(b_0)}(h(b_0) - h(b'_0))e^{i\nu(w)}}{2\sqrt{2(w - x)}} \hat{r}(x_k) \right) + O(R^{-1-\varepsilon}) \\ &= g(b'_0) - g(b_0) + \Re \left(\overline{g(b_0)}(b_0 - b'_0)\pi\hat{r}(x_k) \right) + O(R^{-1-\varepsilon}) \end{aligned}$$

(compare also with Lemma 20). Let $\underline{b}' = (b_1, \dots, b_n)$ with b'_0 substituted for $b_k = b_0$. Then taking into account $g(b_1) - g(b_0) \in \mathbb{R}$, we get

$$\pm \frac{S_{\underline{b}', \underline{w}}(b_0, w)}{S_{\underline{b}, \underline{w}}(b'_0, w)} = 1 + \Re((b'_0 - b_0)\pi\hat{r}(x_k)) + O(R^{-1-\varepsilon})$$

From

$$\langle : \mathcal{O}_1(w_1) \dots \mathcal{O}_1(w_m) \mathcal{O}_{-1}(b_1) \mathcal{O}_{-1}(b_m) : \rangle_{\mathbb{C}} = c \left| \frac{\prod_{i < j} (b_i - b_j)(w_i - w_j)}{\prod_{i \neq j} (b_i - w_j)} \right|^{1/2}$$

we get the following variation when $b_k = b_0$ is replaced with b'_0 :

$$\frac{\langle : \mathcal{O}_1(w_1) \dots \mathcal{O}_1(w_m) \mathcal{O}_{-1}(b_1) \dots \mathcal{O}(b'_0) \dots \mathcal{O}_{-1}(b_m) : \rangle_{\mathbb{C}}}{\langle : \mathcal{O}_1(w_1) \dots \mathcal{O}_1(w_m) \mathcal{O}_{-1}(b_1) \dots \mathcal{O}(b_0) \dots \mathcal{O}_{-1}(b_m) : \rangle_{\mathbb{C}}} = 1 + \Re((b'_0 - b_0)\pi\hat{r}(x_k)) + O(R^{-1-\varepsilon})$$

Consequently, we have:

Theorem 28. *Let $b_1, \dots, b_m, w_1, \dots, w_m$ be marked black (resp. white) vertices on M , with pairwise distances of order $R \gg 1$. Then there is $c_m(\Lambda, \underline{w}) > 0$ such that*

$$\text{Mon}_M(\underline{b}, \underline{w}) = c_m(\Lambda, \underline{w}) \left| \frac{\prod_{i < j} (b_i - b_j)(w_i - w_j)}{\prod_{i \neq j} (b_i - w_j)} \right|^{1/2} (1 + o(1))$$

Again in the case of the square lattice, one may switch colours, so that $c_m(\Lambda, \underline{w}) = c_m(\mathbb{Z}^2)$. Presumably $c_m(\mathbb{Z}^2) = c_1(\mathbb{Z}^2)^m$, though that requires an additional argument.

Let us point out a by-product of the method. In the course of the argument, we have considered trimers of type $\{b_0, w_0, b'_0\}$, three consecutive vertices on the boundary of a face f of M . Then we have for example the following expression for the trimer-monomer correlator:

$$\text{Mon}_M(\{b_0, w_0, b'_0\}, w_1) = \text{Mon}_M(b_0, w_1) |S_{b_0, w_1}(b'_0, w_0)| = \text{Mon}_M(b'_0, w_1) |S_{b_0, w_1}(b'_0, w_0)|$$

and $|S_{b_0, w_1}(b'_0, w_0)| = |g(b'_0) - g(b_0)| (1 + o(1))$ as $|b_0 - w_1| \rightarrow \infty$, where $g(b'_0) - g(b_0)$ depends only on the geometry of the face f :

$$g(b'_0) - g(b_0) = \frac{1}{2} \cot \arg \left(\frac{b'_0 - w_0}{x - w_0} \right) - \frac{1}{2} \cot \arg \left(\frac{b_0 - w_0}{x - w_0} \right) \neq 0$$

where $x = \frac{b'_0 + b_0}{2}$ is the center of f . In the general case, we have

$$\text{Mon}_M(\{b_1, w'_1, b'_1\}, \dots, \{b_m, w'_m, b'_m\}, w_1, \dots, w_m) \sim c_m(\Lambda, \underline{w}) \prod_{i=1}^m \left(\frac{\cot(\theta_i) + \tan(\theta_i)}{2} \right) \left| \frac{\prod_{i < j} (b_i - b_j)(w_i - w_j)}{\prod_{i \neq j} (b_i - w_j)} \right|^{1/2}$$

where $\theta_i \in (0, \frac{\pi}{2})$ is an angle of the right-angled triangle $\{b_i, w'_i, b'_i\}$.

Let us phrase a more general conjecture, following Ciucu. For simplicity we consider the case $M = \mathbb{Z}^2$. An *islet* I is a finite subset of vertices of \mathbb{Z}^2 bounded by a simple loop on $(\mathbb{Z}^2)^\dagger$. Its *charge* is $\varepsilon(I) = |I \cap M_W| - |I \cap M_B|$. The conjecture is that an islet I is a discrete version of a magnetic operator $\mathcal{O}_{\varepsilon(I)}$ in the sense that

$$\text{Mon}_M(I_1 + x_1, \dots, I_n + x_n) \sim c(I_1, \dots, I_n) \prod_{1 \leq i < j \leq n} |x_i - x_j|^{\varepsilon(I_i)\varepsilon(I_j)/2} (1 + o(1))$$

as the pairwise distances $|x_i - x_j|$ go to infinity (we require here $\sum_i \varepsilon(I_i) = 0$).

Lastly let us point out that the surgery argument enables to analyse at essentially no additional cost mixed magnetic-electric correlators, the simplest of which is

$$\langle \mathcal{O}_1(w) \mathcal{O}_{-1}(b) \exp(2i\pi s(\phi(f') - \phi(f))) \rangle$$

where $s \in (0, \frac{1}{2})$ and the pairwise distances between insertions b, w, f', f go to infinity (and $\mathcal{O}_1(w), \mathcal{O}_{-1}(b)$ represent monomer defects). Of some interest are the coincident magnetic-electric operators. At the lattice level, one may consider the above correlator with w on the boundary of f and b on the boundary of f' . For general s , their analysis seems to require additional arguments. Note however that in the coincident case, for $s = \frac{1}{2}$,

$$\langle \mathcal{O}_1(w) \mathcal{O}_{-1}(b) \exp(i\pi(\phi(f') - \phi(f))) \rangle = \pm \mathbb{K}^{-1}(b, w)$$

the asymptotics of which underpin the analysis of all other correlators.

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